 Homework # 3 Solution of honors section problem

Problem 5 $$\frac{y}{x^2+y^2}\,dx - \frac{x}{x^2+y^2}\,dy = 0 \qquad (2)$$

on $\mathbb{R}^2 \setminus (0,0)$

(a) $$P(x,y) = \frac{y}{x^2+y^2} \Rightarrow P_y = \frac{1}{x^2+y^2} - y\frac{2y}{(x^2+y^2)^2} = \frac{x^2+y^2-2y^2}{(x^2+y^2)^2}$$

$$= \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$Q(x,y) = -\frac{x}{x^2+y^2} \Rightarrow Q_x = -\frac{1}{x^2+y^2} + x\frac{2x}{(x^2+y^2)^2} = \frac{2x^2-x^2-y^2}{(x^2+y^2)^2}$$

$$= \frac{x^2-y^2}{(x^2+y^2)^2} \Rightarrow P_y = Q_x$$

(b) (2) is exact on $\mathbb{R}^2\setminus(0,0)$ $\Leftrightarrow$ the vector field $\vec{F} = \langle P, Q \rangle$ is conservative on $\mathbb{R}^2 \setminus (0,0)$ $\Leftrightarrow$ the line integral

$$\oint_C P\,dx + Q\,dy = 0$$ for any closed curve C in $\mathbb{R}^2 \setminus (0,0)$

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Let us show that the last condition does not hold for some closed curve C.

Namely, take C be the unit circle $x^2+y^2=1$ around $(0,0)$:

$$\int_{x^2+y^2=1} \frac{y}{x^2+y^2}\,dx - \frac{x}{x^2+y^2}\,dy = \int_{x^2+y^2=1} \underbrace{y}_{P}\,dx \underbrace{-x}_{Q}\,dy =$$

because the denominators are equal to 1 on the curve of integration

$$\underset{\text{Green theorem}}{=} \iint_{x^2+y^2\le 1} (Q_x - P_y)\,dx\,dy = -2\iint_{x^2+y^2\le 1} dx\,dy =$$

$$= -2 \times (\text{area of } x^2+y^2\le 1) = -2\pi \neq 0$$

Remark Another way is to try to find the potential. If you will do it formally without looking on the geometry of $\mathbb{R}^2\setminus\{0,0\}$ you will arrive to $\varphi(x,y) = \arctan \frac{x}{y}$

(and probably you will not notice anything wrong with this)

(-3-) However, more close look will remind you that arctan is actually a multi-valued function (because the equation $\tan x = a$ has infinite many solution) and $\varphi(x,y)$ is actually equal to the angle of the position vector (x,y) with the y-axis (i.e. $\frac{\pi}{2}$ - the angle component of the polar coordinate) and such φ is not defined continuously on $\mathbb{R}^2 \setminus (0,0)$, because turning around zero (in counter-clock wise direction) will decrease the value of φ by 2π (while making this turn we return to the same point in (x,y)-plane). This matches exactly the result in the previous prove that

$$\int_C P\,dx + Q\,dy = -2\pi$$ for a simple closed path around the origin.