# Solutions of Homework #3, MATH308, Summer 12

## Problem 1

$$ty' + 2y = \sin 2t, \quad t > 0$$

$$\Updownarrow$$

$$y' + \frac{2}{t} y = \frac{\sin 2t}{t}$$

$P(t) = \frac{2}{t}$, $g(t) = \frac{\sin 2t}{t}$

An integrating factor satisfies

$$\mu' = \frac{2}{t}\mu \Rightarrow \quad \mu \text{ can be taken as}$$

$$\mu = e^{\int \frac{2}{t} dt} = e^{2\ln t} = t^2 \Rightarrow$$

$\mu = t^2 \Rightarrow$

$$(\mu y)' = g\mu, \quad \text{i.e}$$

$$(t^2 y)' = \frac{\sin 2t}{t} \cdot t^2 = t \sin 2t \Rightarrow$$

$$t^2 y = \int t \sin 2t \, dt + C$$

To calculate $\int t \sin 2t \, dt$ use the integration by parts

$$\int t \sin 2t \, dt = -\frac{1}{2} t \cos 2t + \frac{1}{2}\int \cos 2t \, dt =$$

$$= -\frac{1}{2} t \cos 2t + \frac{1}{4}\sin 2t \Rightarrow$$

$$t^2 y = -\frac{1}{2} t \cos 2t + \frac{1}{4}\sin 2t + C \Rightarrow$$

Gen. solution is $\boxed{y(t) = -\frac{\cos 2t}{t} + \frac{1}{4t^2}\sin 2t + \frac{C}{t^2}}$

No matter what is C:

$$\boxed{y(t) \xrightarrow[t\to+\infty]{} 0}$$

Problem 2 (a) $y' - 10y = t^2e^{9t} \Rightarrow p = -10,\ g = t^2e^{9t}$

An integrating factor satisfies

$\mu' = -10\mu \Rightarrow$ we can take $\mu = e^{-10t}$

$$\Downarrow$$

$(\mu y)' = g\mu \Leftarrow (e^{-10t}y)' = t^2e^{9t}e^{-10t} = t^2e^{-t} \Rightarrow$

$e^{-10t}y = \int t^2e^{-t}dt + C$

Let us evaluate $\int t^2e^{-t}dt$ using integration by parts twice.

$\int t^2e^{-t}dt = -t^2e^{-t} + 2\int te^{-t}dt = -t^2e^{-t} - 2te^{-t} +$

$+2\int e^{-t}dt = -t^2e^{-t} - 2te^{-t} - 2e^{-t} = -(t^2+2t+2)e^{-t}$

$$\Downarrow$$

$e^{-10t}y(t) = -(t^2+2t+2)e^{-t} + C \Rightarrow$

$y(t) = -(t^2+2t+2)e^{9t} + Ce^{10t}$

Now find the solution with $y(0) = a$:

$a = y(0) = -2 + C \Rightarrow C = a+2 \Rightarrow$

$$\boxed{y(t) = -(t^2+2t+2)e^{9t} + (a+2)e^{10t}}$$

b) The sign of $a+2$ is crucial

If $a+2<0$ ($\Leftrightarrow a<-2$) then

$$\lim_{t\to+\infty} y(t) = \lim_{t\to+\infty}\Big(\underbrace{-(t^2+2t+2)e^{9t}}_{\downarrow t\to+\infty \atop -\infty} + \underbrace{(a+2)e^{10t}}_{\downarrow t\to-\infty \atop -\infty}\Big) = -\infty$$

If $a+2=0$ ($\Leftrightarrow a=-2$), then in the same way $\lim\limits_{t\to+\infty} y(t) = -\infty$

If $a+2>0$ ($\Leftrightarrow a>-2$), then

$$\lim_{t\to+\infty} y(t) = \lim_{t\to+\infty}\left(-(t^2+2t+2)e^{9t} + (a+2)e^{10t}\right) =$$

$$= \lim_{t\to+\infty} \underbrace{e^{9t}}_{\downarrow t\to+\infty \atop +\infty} \underbrace{\big(\overset{>0}{(a+2)}e^{t} - (t^2+2t+2)\big)}_{\downarrow t\to+\infty \atop +\infty} = +\infty$$

$\Downarrow$

The value $a_0$ from which the transition from one type of the behavior to another type occurs is equal to $-2$, $\boxed{a_0=-2}$

c) As was already mentioned in item b

If $a=-2$ then $\boxed{\lim\limits_{t\to+\infty} y(t) = -\infty}$