Homework Assignment 4 in Differential Equations, MATH308-Fall 2016

due September 26, 2016

Topics covered : *exact equations: the method of integrating factor (sections 2.6); Wronskian, fundamental set of solutions of linear homogeneous equations of second order. linear homogeneous equations of second order with constant coefficients: the case of two distinct real roots of the characteristic polynomial (corresponds to sections 3.2, and 3.1 in the textbook).*

1. In both items below one can find an integral factor which depends either on x or on y to make it exact. Find this integrating factor and then solve the equation (find the general solution:
   (a) $(x^2 + y^2 + x)dx + ydy = 0$;
   (b) $y\sqrt{1+y^2}dx = (y - x\sqrt{1+y^2})dy$.
2. The differential equation $f(x)\frac{dy}{dx} + x^2 + y = 0$ is known to have an integrating factor $\mu(x) = x$. Find all possible functions $f(x)$.
3. Show that $y(t) = t^2$ can never be a solution of the differential equation

   $$y'' + p(t)y' + q(t)y = 0$$

   if the functions $p(t)$ and $q(t)$ are continuous in an open interval containing $t = 0$.
4. (a) Calculate the Wronskian of the the pair of the functions $e^{\alpha t}\cos(\beta t)$, $e^{\alpha t}\sin(\beta t)$.
   (b) Assume that $y_1(t)$ and $y_2(t)$ are two solutions of the equation

   $$y'' + p(t)y' + q(t)y = 0 \tag{1}$$

   on the interval $(-1, 1)$, where the functions $p(t)$ and $q(t)$ are continuous on the same interval. It is known that $y_1(0) = 3$, $y_1'(0) = 5$, $y_2(0) = -6$, and $y_2'(0) = -9$. Is it true that the general solution of (1) is $C_1y_1(t) + C_2y_2(t)$? Justify your answer.
   (c) Answer the same question as in the previous item if $y_1(0) = 3$, $y_1'(0) = 5$, $y_2(0) = -6$, but $y_2'(0) = -10$. Justify your answer. How y_1 and y_2 are related in this case?
5. (a) Find the solution of the initial value problem

   $$20y'' - 17y' + 3y = 0, \quad y(0) = \alpha,\ y'(0) = \frac{1}{4}, \tag{2}$$

   where α is a parameter.
   (b) Determine all values of α, if any, for which the solution of the initial value problem (2) tends to $+\infty$ as $t \to +\infty$.
6. (**bonus-30 points**) Read the formulation and the proof of Abel's Theorem (Theorem 3.2.7 in 10th edition and Theorem 3.2.6 in ninth edition). Then, using this theorem, solve the following problem: Let y_1 and y_2 be solutions of *Bessel's equation*

   $$t^2y'' + ty' + (t^2 - n^2)y = 0$$

   on the interval $t > 0$ with $y_1(1) = 2$, $y_1'(1) = -3$, $y_2(1) = -4$, $y_2'(1) = 7$. Compute $W(y_1, y_2)(3)$.