Problem 1

(a) $(x^2+y^2+x)dx + y\,dy = 0$

$P = x^2+y^2+x \qquad P_y = 2y \qquad P_y \neq Q_x$

$Q = y \qquad Q_x = 0$

$\frac{P_y - Q_x}{Q} = \frac{2y}{y} = 2 \rightarrow$ does not depend on $y \Rightarrow$ we can find the integrating factor

μ depends on x only

$\frac{d}{dx}\mu = 2\mu \Rightarrow$ we can take $\mu = e^{2x}$

$e^{2x}(x^2+y^2+x)\,dx + e^{2x}y\,dy = 0$ .. This equation is exact. Find potential

$$\begin{cases} \varphi_x = e^{2x}(x^2+y^2+x) \\ \varphi_y = e^{2x}y \Rightarrow \varphi = \frac{1}{2}e^{2x}y^2 + h(x) \end{cases}$$

$\Rightarrow$ (substituting to the first equation)

$\varphi_x = \cancel{e^{2x}y^2} + h'(x) = e^{2x}x^2 + \cancel{e^{2x}y^2} + e^{2x}x \Rightarrow$

$h'(x) = e^{2x}(x^2+x) \Rightarrow$

$h(x) = \int \underbrace{e^{2x}}_{u'}\underbrace{(x^2+x)}_{v}dx \underset{\text{by parts}}{=} \frac{1}{2}e^{2x}(x^2+x) - \frac{1}{2}\int \underbrace{e^{2x}}_{u'}\underbrace{(2x+1)}_{v}dx =$

$u = \frac{1}{2}e^{2x},\ v' = 2x+1 \qquad u = \frac{1}{2}e^{2x}\ v' = 2$

$= \frac{1}{2}e^{2x}(x^2+x) - \frac{1}{4}e^{2x}(2x+1) + \frac{1}{4}\underbrace{\int e^{2x}\,2\,dx}_{e^{2x}} =$

$\frac{1}{2}e^{2x}\left(x^2 + \cancel{x} - \cancel{x} - \cancel{\tfrac{1}{2}} + \cancel{\tfrac{1}{2}}\right) = \frac{1}{2}x^2e^{2x}$

(we do not need to add a constant, because we are looking for some potential)

$\Rightarrow \quad \varphi = \frac{1}{2}e^{2x}y^2 + \frac{1}{2}x^2e^{2x} = \frac{1}{2}e^{2x}(x^2+y^2) \Rightarrow$

The general solution is $\boxed{\frac{1}{2}e^{2x}(x^2+y^2) = C}$

Problem 1

(b) $y\sqrt{1+y^2}\,dx = (y - x\sqrt{1+y^2})\,dy$

$$y\sqrt{1+y^2}\,dx - (y - x\sqrt{1+y^2})\,dy = 0$$

$P = y\sqrt{1+y^2}$ $\qquad P_y = \sqrt{1+y^2} + \frac{y^2}{\sqrt{1+y^2}}$ $\qquad P_y \neq Q_x$

$Q = -(y - x\sqrt{1+y^2})$ $\qquad Q_x = \sqrt{1+y^2}$

$$\frac{Q_x - P_y}{P} = -\frac{\frac{y^2}{\sqrt{1+y^2}}}{y\sqrt{1+y^2}} = -\frac{y}{1+y^2}$$ → does not depend on x ⇒ we

can find the integrating factor μ depending on y only by solving

$$\frac{d\mu}{dy} = -\frac{y}{1+y^2}\mu \Leftrightarrow \underset{\text{we can take}}{\mu} = e^{-\int \frac{y}{1+y^2}dy} = e^{-\frac{1}{2}\ln(1+y^2)} =$$

$= \frac{1}{\sqrt{1+y^2}}$ ⇒ after multiplying by the integrating factor

we have

$$y\,dx - \left(\frac{y}{\sqrt{1+y^2}} - x\right)dy = 0$$

This equation is exact. Find potential

$\varphi_x = y \Rightarrow \varphi = xy + h(y) \Rightarrow \varphi_y = \cancel{x} + h'(y) = \cancel{x} - \frac{y}{\sqrt{1+y^2}}$

$\varphi_y = x - \frac{y}{\sqrt{1+y^2}}$ $\qquad h'(y) = -\frac{y}{\sqrt{1+y^2}} \Rightarrow$

$$h(y) = -\int \frac{y}{\sqrt{1+y^2}}dy = -\frac{1}{2}\int u^{-1/2}du = -\frac{1}{2}\frac{u^{1/2}}{1/2} = -u^{1/2} =$$

$u = 1+y^2 \Rightarrow du = 2y\,dy$ $\qquad = -\sqrt{1+y^2} \Rightarrow$

φ can be taken as $\varphi = xy - \sqrt{1+y^2} \Rightarrow \boxed{xy - \sqrt{1+y^2} = C}$ is the general solution.

Problem 2

If $\mu(x)=x$ is an integrating factor then the equation

$$xf(x)\frac{dy}{dx}+x^3+xy=0 \quad \text{must be exact}$$

$$\underbrace{(x^3+xy)}_{P}dx+\underbrace{(xf(x))}_{Q}dy=0$$

So $P_y = x$

$Q_x = \frac{d}{dx}(xf(x))$ $\Rightarrow$ The condition for exactness is

$$\frac{d}{dx}(xf(x)) = x \Leftrightarrow$$

$$xf(x) = \frac{1}{2}x^2+C \Rightarrow$$

$$\boxed{f(x) = \frac{1}{2}x+\frac{C}{x}}$$

Problem 3

Elementary solution (without any uniqueness theorem)

If $y(t)=t^2$ is a solution then plugging it into equation

we get $\underbrace{2}_{y''}+p(t)t+q(t)t^2=0$

Then for $t=0$ $\quad 2=0 \Rightarrow$ contradiction

Another solution based on uniqueness theorem

Assume that $y_1(t)=t^2$ is a solution of $y''+p(t)y'+p(t)y=0$

so that $y_1(0)=0$ and $y_1'(0)=0$

Note that $y_2(t)=0$ is also a solution and

$y_2(0)=0$ and $y_2'(0)=0$ i.e. the same initial condition $\Rightarrow$ from uniqueness theorem $y_1(t) \equiv y_2(t)$ for all t

In an open interval around 0 but $t^2 \neq 0$ in an open interval of $t=0$ $\Rightarrow$ contradiction.

Problem 4

(a) $y_1 = e^{\alpha t}\cos\beta t \qquad y_2 = e^{\alpha t}\sin\beta t$

$y_1' = \alpha e^{\alpha t}\cos\beta t - e^{\alpha t}\beta\sin\beta t \qquad y_2' = \alpha e^{\alpha t}\sin\beta t + \beta e^{\alpha t}\cos\beta t$

$$W(y_1,y_2)(t) = \begin{vmatrix} e^{\alpha t}\cos\beta t & e^{\alpha t}\sin\beta \\ \alpha e^{\alpha t}\cos\beta t - e^{\alpha t}\beta\sin\beta t & \alpha e^{\alpha t}\sin\beta t + \beta e^{\alpha t}\cos\beta t \end{vmatrix} =$$

$$= e^{\alpha t}\cos\beta t\left(\alpha e^{\alpha t}\sin\beta t + \beta e^{\alpha t}\cos\beta t\right) - \left(\alpha e^{\alpha t}\cos\beta t - e^{\alpha t}\beta\sin\beta t\right)e^{\alpha t}\sin\beta t =$$

$$= \cancel{\alpha e^{2\alpha t}\cos\beta t\sin\beta t} + \beta e^{2\alpha t}\cos^2\beta t - \cancel{\alpha e^{2\alpha t}\cos\beta t\sin\beta t} + \beta e^{2\alpha t}\sin^2\beta t =$$

$$= \beta e^{2\alpha t}\underbrace{(\cos^2\beta t + \sin^2\beta t)}_{1} = \boxed{\beta e^{2\alpha t}}$$

(b) $W(y_1,y_2)(0) = \begin{vmatrix} 3 & -6 \\ 5 & -9 \end{vmatrix} = -27+30 = 3 \neq 0 \Rightarrow$

$y_1(t)$ and $y_2(t)$ is a fundamental set of solutions $\Rightarrow$

the general solution is

$$y(t) = C_1 y_1(t) + C_2 y_2(t)$$

(c) $W(y_1,y_2)(0) = \begin{vmatrix} 3 & -6 \\ 5 & -10 \end{vmatrix} = -30+30 = 0 \Rightarrow$

$C_1 y_1(t) + C_2 y_2(t)$ is <u>not</u> a general solution and $y_2(t) = c\,y_1(t)$ for any t

since $y_2(0) = -2y_1(0)$ then $c=-2 \Rightarrow$

$$\boxed{y_2(t) = -2\,y_1(t).}$$

Problem 5

(a) $20y'' - 17y' + 3y = 0$

$y(0) = \alpha$

$y'(0) = \frac{1}{4}$

The characteristic equation is

$$20r^2 - 17r + 3 = 0$$

$$D = (17)^2 - 4\cdot 20\cdot 3 = 289 - 240 = 49 = 7^2$$

$$r_1 = \frac{17+7}{40} = \frac{24}{40} = \frac{3}{5}$$

$$r_2 = \frac{17-7}{40} = \frac{10}{40} = \frac{1}{4}$$

$\Rightarrow$ the general solution is

$$y(t) = c_1 e^{\frac{3}{5}t} + c_2 e^{\frac{1}{4}t}$$

$y(0) = \alpha \Rightarrow$
$y'(0) = \frac{1}{4}$

$$\begin{cases} c_1 + c_2 = \alpha & (\text{Eq } 1) \\ \frac{3}{5}c_1 + \frac{1}{4}c_2 = \frac{1}{4} & (\text{Eq } 2) \end{cases}$$

Eliminate c_2

$(\text{Eq }1) - (4\,\text{Eq }2)$:

$$\underbrace{\left(1 - \frac{12}{5}\right)}_{-\frac{7}{5}} c_1 = \alpha - 1 \Rightarrow$$

$$c_1 = \frac{5}{7}(1-\alpha)$$

From (Eq 1) $c_2 = \alpha - c_1 = \alpha - \frac{5}{7} + \frac{5}{7}\alpha = \frac{12}{7}\alpha - \frac{5}{7} \Rightarrow$

$$\boxed{y(t) = \frac{5}{7}(1-\alpha)e^{\frac{3}{5}t} + \left(\frac{12}{7}\alpha - \frac{5}{7}\right)e^{\frac{1}{4}t}}$$

(c) $y(t) = \frac{5}{7}(1-\alpha)e^{\frac{3}{5}t} + \left(\frac{12}{7}\alpha - \frac{5}{7}\right)e^{\frac{1}{4}t} = e^{\frac{3}{5}t}\left(\frac{5}{7}(1-\alpha) + \left(\frac{12}{7}\alpha - \frac{5}{7}\right)\underbrace{e^{\left(\frac{1}{4}-\frac{3}{5}\right)t}}_{\substack{\downarrow\, t\to+\infty \\ 0}}\right)$

Therefor if $\frac{5}{7}(1-\alpha) > 0$ $(\Leftrightarrow \alpha < 1)$ then $y(t) \xrightarrow[t\to+\infty]{} +\infty$ ✓

If $\frac{5}{7}(1-\alpha) < 0$ $(\Leftrightarrow \alpha > 1)$ then $y(t) \xrightarrow[t\to-\infty]{} -\infty$ ✗

If $\frac{5}{7}(1-\alpha) = 0$ $(\Leftrightarrow \alpha = 1)$ then $y(t) = \underbrace{\left(\frac{12}{7} - \frac{5}{7}\right)}_{1}e^{\frac{1}{4}t} \xrightarrow[t\to+\infty]{} +\infty$ ✓

Combining all these case we get the answer $\alpha \leq 1$

Problem 6 (bonus)

$$t^2y'' + ty' + (t^2 - n^2)y = 0$$

$$y'' + \frac{1}{t}y' + \left(\frac{t^2-n^2}{t}\right)y = 0$$

$$p(t) = \frac{1}{t}$$

$\Rightarrow$ according to the Abel theorem the Wronskian $W(t) = W(y_1, y_2)(t)$ satisfies the differential equation

$$W' = -\frac{1}{t}W \Rightarrow$$

$$W(t) = Ce^{-\int \frac{1}{t}dt} = Ce^{-\ln t} = \frac{C}{t}$$

By assumptions $W(1) = \begin{vmatrix} y_1(1) & y_2(1) \\ y_1'(1) & y_2'(1) \end{vmatrix} = \begin{vmatrix} 2 & -4 \\ -3 & 7 \end{vmatrix} = 14 - 12 = 2$

$\Rightarrow 2 = W(1) = \frac{C}{1} \Rightarrow C = 2 \Rightarrow W(t) = \frac{2}{t} \Rightarrow \boxed{W(3) = \frac{2}{3}}$