Homework # 4 MATH 308-505 Solutions

Problem 1 $\quad (t^2+7t+10)y' + (t-2)y = te^t \quad (1)$

$$t^2+7t+10 = (t+2)(t+5)$$

(1) $\Leftrightarrow$ $$y' + \frac{t-2}{(t+2)(t+5)}y = \frac{te^t}{(t+2)(t+5)}$$

$p(t) = \frac{t-2}{(t+2)(t+5)}$ has discontinuity at $t=-2$ and $t=-5$

$g(t) = \frac{te^t}{(t+2)(t+5)}$ has discontinuity at the same points

(a) $t_0 = -6 < -5 \Rightarrow$ the solution is certain to exist for $\boxed{t < -5}$

(b) $t_0 = -3$, $-5 < -3 < -2 \Rightarrow$ the solution is certain to exist for $\boxed{-5 < t < -2}$

(c) $t_0 = 0 > -2 \Rightarrow$ the solution is certain to exist for $\boxed{t > -2}$

Problem 2

$$\left(ax^3y-5e^y\right)\frac{dy}{dx}=\left(x^2y^2+\sin x\right)$$

$$\left(x^2y^2+\sin x\right)dx-\left(ax^3y-5e^y\right)dy=0$$

$$\begin{aligned} P&=x^2y^2+\sin x \\ Q&=-\left(ax^3y-5e^y\right)= \\ &=-ax^3y+5e^y \end{aligned} \Rightarrow \begin{aligned} P_y&=2x^2y \\ Q_x&=-3ax^2y \end{aligned}$$

$$P_y\equiv Q_x \iff 2x^2y=-3ax^2y \Rightarrow \boxed{a=-\frac{2}{3}}$$

If $a=-\frac{2}{3}$, then $Q=\frac{2}{3}x^3y+5e^y$

Find the potential:

$$\begin{cases} \varphi_x=x^2y^2+\sin x \\ \varphi_y=\frac{2}{3}x^3y+5e^y \end{cases} \Rightarrow \begin{aligned} \varphi&=\int\left(x^2y^2+\sin x\right)dx+h(y)= \\ &=\frac{1}{3}x^3y^2-\cos x+h(y) \end{aligned}$$

Substituting to the second equation:

$$\cancel{\frac{2}{3}x^3y}+h'(y)=\cancel{\frac{2}{3}x^3y}+5e^y \Rightarrow h'(y)=5e^y \Rightarrow h(y)=5e^y+C$$

We can take $C=0$ $\Rightarrow$

$$\varphi=\frac{1}{3}x^3y^2-\cos x+5e^y \Rightarrow \text{the general}$$

solution is

$$\frac{1}{3}x^3y^2-\cos x+5e^y=C$$

$y(\pi)=0 \Rightarrow$

$$C=-\cos\pi+5=6 \Rightarrow$$

$\Rightarrow$ the require solution is $\boxed{\frac{1}{3}x^3y^2 - \cos x + 5e^y = 6}$

Problem 3 $(3xy - y^2)dx + x(x-y)dy = 0$

$P = 3xy - y^2$ $\qquad P_y = 3x - 2y$ $\qquad P_y \neq Q_x \Rightarrow$ not exact

$Q = x^2 - xy$ $\qquad Q_x = 2x - y$

However $\frac{P_y - Q_x}{Q} = \frac{3x - 2y - 2x + y}{x(x-y)} = \frac{\cancel{x-y}}{x\cancel{(x-y)}} = \frac{1}{x}$

depend on x only $\Rightarrow$ we can look for the integrating factor μ depending on x only, solving

$$\mu_x = \frac{1}{x}\mu \Rightarrow \text{one can take } \mu = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

Multiplying by $\mu = x$ our equation we get an exact equation

$$(3x^2y - y^2x)\,dx + (x^3 - x^2y)\,dy$$

$P = 3x^2y - y^2x$ $\qquad$ Now $P_y = 3x^2 - 2xy$, $Q_x = 3x^2 - 2xy$

$Q = x^3 - x^2y$ $\qquad P_y = Q_x$

Find the potential $\qquad$ $\to$ this is not necessary just check

$$\begin{cases} \varphi_x = 3x^2y - y^2x \Rightarrow \varphi = x^3y - \frac{y^2x^2}{2} + h(y) \\ \varphi_y = x^3 - x^2y \qquad \varphi_y = x^3 - x^2y + h'(y) = \end{cases}$$

$$= x^3 - x^2y \Rightarrow h'(y) = 0 \Rightarrow h(y) \equiv c$$

Take $c = 0 \Rightarrow \varphi = x^3y - \frac{x^2y^2}{2} \Rightarrow$ gen. solution is

$$x^3y - \frac{x^2y^2}{2} \equiv \text{const}$$