Homework Assignment 4 in MATH308-Spring 2016, Honors section

due February 23, 2017

Topics covered : *Wronskian, fundamental set of solutions of linear homogeneous equations of second order. linear homogeneous equations of second order with constant coefficients (sections 3.1-3.4).*

1. (a) Calculate the Wronskian of the the pair of the functions $\sin^2 t$, $1-\cos 2t$.

   (b) Assume that $y_1(t)$ and $y_2(t)$ are two solutions of the equation

$$y'' + p(t)y' + q(t)y = 0 \qquad (1)$$

   on the interval $(-1, 1)$, where the functions $p(t)$ and $q(t)$ are continuous on the same interval. It is known that $y_1(0) = 2$, $y_1'(0) = -4$, $y_2(0) = -3$, and $y_2'(0) = 5$. Is it true that the general solution of (1) is $C_1y_1(t) + C_2y_2(t)$? Justify your answer.

   (c) Answer the same question as in the previous item if $y_1(0) = 2$, $y_1'(0) = -4$, $y_2(0) = -3$, but $y_2'(0) = 6$. Justify your answer. How y_1 and y_2 are related in this case (namely, express $y_2(t)$ in terms of $y_1(t)$, if possible?

   (d) Assume again that $y_1(t)$ and $y_2(t)$ are solutions of (1) on the interval $(-1, 1)$. Prove that if $y_1(t)$ and $y_2(t)$ achieve a maximum or minimum at the same point, then they cannot form a fundamental set of solutions on this interval.

   (e) Suppose that the Wronskian of any two solutions of (1) is constant in time. Prove that $p(t) \equiv 0$. (Hint: Use the Abel theorem)

   (f) (**bonus-20 points**) Assume that $y_1(t)$ and $y_2(t)$ form a fundamental set of solutions of (1) on the interval $(-1, 1)$. Assume also that for some t_1 and t_2 in $(-1, 1)$ with $t_1 < t_2$ we have: $y_2(t_1) = 0$, $y_2(t_2) = 0$, and $y_2(t) \neq 0$ for all t in the interval (t_1, t_2). Show that there exists a unique c in the interval (t_1, t_2) such that $y_1(c) = 0$. (Hint: Differentiate the quantity $\frac{y_2}{y_1}$)

2. Consider differential equations $\frac{1}{3}y'' - y' + \alpha y = 0$, where α is a parameter.

   (a) In each of the following three cases find the general solution of the differential equation corresponding to the given α, if:
   i) $\alpha = \frac{5}{12}$; ii) $\alpha = \frac{3}{4}$; iii) $\alpha = \frac{13}{12}$.
   (if the roots of the characteristic equation are complex, please find the real-valued general solution, i.e. the answer in the form $C_1e^{rt} + C_2e^{\bar{r}}$, where r is complex, will not be accepted)).

   (b) For the case of item (a) i), i.e. when $\alpha = \frac{5}{12}$, solve the following problems:
      i. find the solution satisfying the initial condition $y(0) = -2$, $y'(0) = V$, where V is a parameter;
      ii. find all parameters V such that the solution $y(t)$ obtained in the previous subitem for such V satisfies: $y(t) < 0$ for any $t \geq 0$.

   (c) For the case of item (a) iii), i.e. when $\alpha = \frac{13}{12}$ solve the following problems:
      i. Find the solution of the equation with the initial conditions $y(\frac{\pi}{2}) = -4e^{\frac{3\pi}{4}}$, $y'(\frac{\pi}{2}) = -9e^{\frac{3\pi}{4}}$. Describe the behavior of the solution as $t \to -\infty$;
      ii. Determine λ, $\mu > 0$, $R > 0$ and $\delta \in [0, 2\pi)$ so that the solution obtained in the previous item can be written in the form $e^{\lambda t}R\cos(\mu t - \delta)$ (you can use a calculator to determine an approximate value of δ). Then sketch the graph of this solution.

3. Consider the equation $y'' + (1-\alpha)y' + (2\alpha-3)(4-3\alpha)y = 0$, where α is a parameter.

   (a) Determine the values of α , if any, for which all solutions tend to zero as $t \to +\infty$.

   (b) Determine the values of α , if any, for which all nonzero solutions become unbounded as $t \to +\infty$.

   Hint: It is more efficient here to use the Vieta theorem for roots of quadratic equation instead of the quadratic formula, but be careful on the signs.