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Homework # 4 Solutions, Regular part

Problem 1

(a) $W(\sin^2 t, 1-\cos 2t) = \begin{vmatrix} \sin^2 t & 1-\cos 2t \\ \underbrace{2\sin t\cos t}_{\sin 2t} & 2\sin 2t \end{vmatrix} =$

$$= \begin{vmatrix} \frac{1-\cos 2t}{2} & 1-\cos 2t \\ \sin 2t & 2\sin 2t \end{vmatrix} = (1-\cos 2t)\sin 2t - (1-\cos 2t)\sin 2t$$

$\sin^2 t = \frac{1-\cos 2t}{2}$

$$= \boxed{0}$$

(b) $W(y_1, y_2)(0) = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix} = \begin{vmatrix} 2 & -3 \\ -4 & 5 \end{vmatrix} = 10-12 =$

$= -2 \neq 0 \Rightarrow y_1(t)$ and $y_2(t)$ is a fundamendal set of solutions $\Rightarrow C_1 y_1(t) + C_2 y_2(t)$ is the general solution

(c) $W(y_1, y_2)(0) = \begin{vmatrix} 2 & -3 \\ -4 & 6 \end{vmatrix} = 12-12 = 0 \Rightarrow$

$y_2(t) = C y_1(t)$. In pardicular $y_2(0) = C y_1(0)$, i.e

$-3 = C \cdot 2 \Rightarrow C = -\frac{3}{2} \Rightarrow \boxed{y_2(t) = -\frac{3}{2} y_1(t)}$

(d) If $y_1(t)$ and $y_2(t)$ attains maximum or minimum at the same point $t = t_0$ then $y_1'(t_0) = y_2'(t_0) = 0 \Rightarrow$

$$W(y_1, y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ 0 & 0 \end{vmatrix} = 0$$

$\Rightarrow y_1$ & y_2 do not form a fundamendal set of solutions

(e) By Abel's Theorem if $W(t) = W(y_1, y_2)(t)$

then $W'(t) = -p(t)W(t)$ $(*)$

Let $y_1(t)$ & $y_2(t)$ is a fundamental set of solution

then $W(t) \equiv C \neq 0$ $\Rightarrow$ substituting into the Abel's

equation $(*)$ we have

$$0 = -p(t) \cdot C$$

(here we use that $W'(t) \equiv 0$) $\Rightarrow$ since $C \neq 0$

we have that $p(t) \equiv 0$

Problem 2 (a) Characteristic equation is

$$\frac{1}{3}r^2 - r + \alpha = 0$$

$$D = 1 - \frac{4}{3}\alpha$$

(i) If $\alpha = \frac{5}{12}$ then $D = 1 - \frac{4}{3} \cdot \frac{5}{12_3} = 1 - \frac{5}{9} = \frac{4}{9} \Rightarrow \sqrt{D} = \frac{2}{3} \Rightarrow$

$r_1 = \dfrac{1 + \frac{2}{3}}{2 \cdot \frac{1}{3}} = \dfrac{5/3}{2/3} = \dfrac{5}{2}$

$r_2 = \dfrac{1 - \frac{2}{3}}{2 \cdot \frac{1}{3}} = \dfrac{1/3}{2/3} = \dfrac{1}{2}$

$r_1 \neq r_2$ real $\Rightarrow$ gen. solution is

$y(t) = C_1 e^{5/2 t} + C_2 e^{1/2 t}$

(ii) if $\alpha = \frac{3}{4}$ then $D = 1 - \frac{4}{3} \cdot \frac{3}{4} = 1 - 1 = 0 \Rightarrow$ repeated roots

$r_1 = r_2 = \dfrac{1}{2 \cdot \frac{1}{3}} = \dfrac{3}{2} \Rightarrow$

gen. solution is $y(t) = (C_1 + C_2 t)e^{3/2 t}$

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(iii) $\alpha = \frac{13}{h} \Rightarrow D = 1 - \frac{4}{3} \cdot \frac{13}{12} = 1 - \frac{13}{9} = -\frac{4}{9} < 0 \Rightarrow \sqrt{D} = \frac{2}{3}i$

$$r_{1,2} = \frac{1 \pm \frac{2}{3}i}{2\frac{1}{3}} = \frac{3}{2} \pm i \Rightarrow \lambda = \frac{3}{2}, \mu = 1 \Rightarrow$$

general solution is $e^{3/2 t}(C_1 \cos t + C_2 \sin t)$

(B) (i)

$y(t) = C_1 e^{5/2 t} + C_2 e^{t/2} \Rightarrow y(0) = -2 \Leftrightarrow \begin{cases} C_1 + C_2 = -2 & (Eq\,1) \\ \frac{5}{2}C_1 + \frac{C_2}{2} = V & (Eq\,2) \end{cases}$

$y'(t) = 5/2\, C_1 e^{5/2 t} + \frac{C_2}{2} e^{t/2} \Rightarrow y'(0) = V \Leftrightarrow$

Eliminate C_2: $(Eq\,1) - 2(Eq\,2) \Rightarrow C_1 - 5C_1 = -2 - 2V \Rightarrow$

$-4C_1 = -2 - 2V \Leftrightarrow \boxed{C_1 = \frac{V+1}{2}} \overset{Eq\,1}{\Rightarrow} C_2 = -2 - \frac{V+1}{2} =$

$= \frac{-4-V-1}{2} = -\frac{V+5}{2}$

$$\Rightarrow y(t) = \frac{V+1}{2} e^{5/2 t} - \frac{V+5}{2} e^{t/2}$$

(ii) $y(t) = \frac{V+1}{2} e^{5/2 t} - \frac{V+5}{2} e^{t/2} = \left(\frac{V+1}{2} e^{2t} - \frac{V+5}{2}\right)\underbrace{e^{t/2}}_{>0}$

The sign of $\frac{V+1}{2}$ is crucial: 1) If $\frac{V+1}{2} < 0$, i.e. $V < -1$

then $\frac{V+1}{2} e^{2t} + \frac{V+5}{2}$ is decreasing and ~~equal~~

to -2 at $t=0 \Rightarrow$ negative for all $t \geq 0 \Rightarrow$

$y(t) < 0$ for all $t \geq 0$

2) If $\frac{V+1}{2} = 0$, i.e. $V = -1$ then $y(t) = -2e^{t/2} < 0$, i.e.

$V = -1$ is also contained in the answer

3) If $\frac{V+1}{2} > 0$, i.e. $V > -1$ then $\lim y(t) \to +\infty \Rightarrow y(t)$ is not negative for all $t \geq 0$

Combining all 3 cases we have that the answer to the question is $\boxed{r \le -1}$

(c) (i) $y(t) = C_1 e^{3/2 t}\cos t + C_2 e^{3/2 t}\sin t$

$y'(t) = \frac{3}{2}C_1 e^{3/2 t}\cos t - C_1 e^{3/2 t}\sin t + \frac{3}{2}C_2 e^{3/2 t}\sin t + C_2 e^{3/2 t}\cos t$

$y(\frac{\pi}{2}) = -4e^{3\pi/4} \Rightarrow C_2 e^{3\pi/4} = -4e^{3\pi/4} \Rightarrow C_2 = -4$

$y'(\frac{\pi}{2}) = -9e^{3\pi/4} \Rightarrow -C_1 \cancel{e^{3\pi/4}} + \frac{3}{2}C_2 \cancel{e^{3\pi/4}} = -9\cancel{e^{3\pi/4}} \Rightarrow$

$-C_1 + \underbrace{\frac{3}{2}\underset{-4}{C_2}}_{-6} = -9 \Rightarrow -C_1 - 6 = -9 \Rightarrow$
$C_1 = 3$

$\Rightarrow \boxed{y(t) = e^{3/2 t}(3\cos t - 4\sin t)}$, $\lim_{t\to-\infty} y(t) = 0$

(because $|y(t)| < 7e^{3/2 t} \underset{t\to-\infty}{\to} 0$)

(ii) $C_1 = 3$, $C_2 = -4$

$R = \sqrt{C_1^2 + C_2^2} = \sqrt{16+9} = \boxed{5}$

$\cos\delta = \frac{C_1}{R} = \frac{3}{5}$ δ is the forth quadrant

$\sin\delta = \frac{C_2}{R} = -\frac{4}{5}$

$\boxed{\delta = \arctan(-\frac{4}{3}) = -\arctan\frac{4}{3} \approx -0.92\text{ rad} \approx -53.13^\circ}$

$\boxed{\lambda = 3/2,\ \mu = 1}$

$\Rightarrow y(t) = e^{3/2 t} 5\cos(t - \delta) = e^{3/2 t} 5\cos(t + \arctan\frac{4}{3})$

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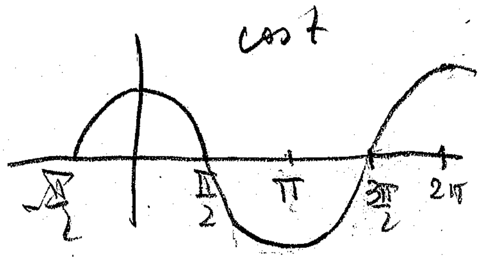
Sketch

Use the method of parent functions using representation

$y(t) = -e^{3/2 t} 5\cos(t - \arctan \frac{3}{4})$

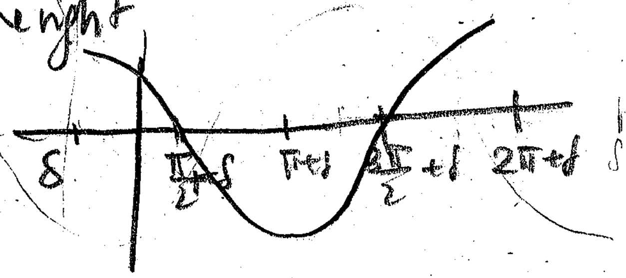
Start with cos t



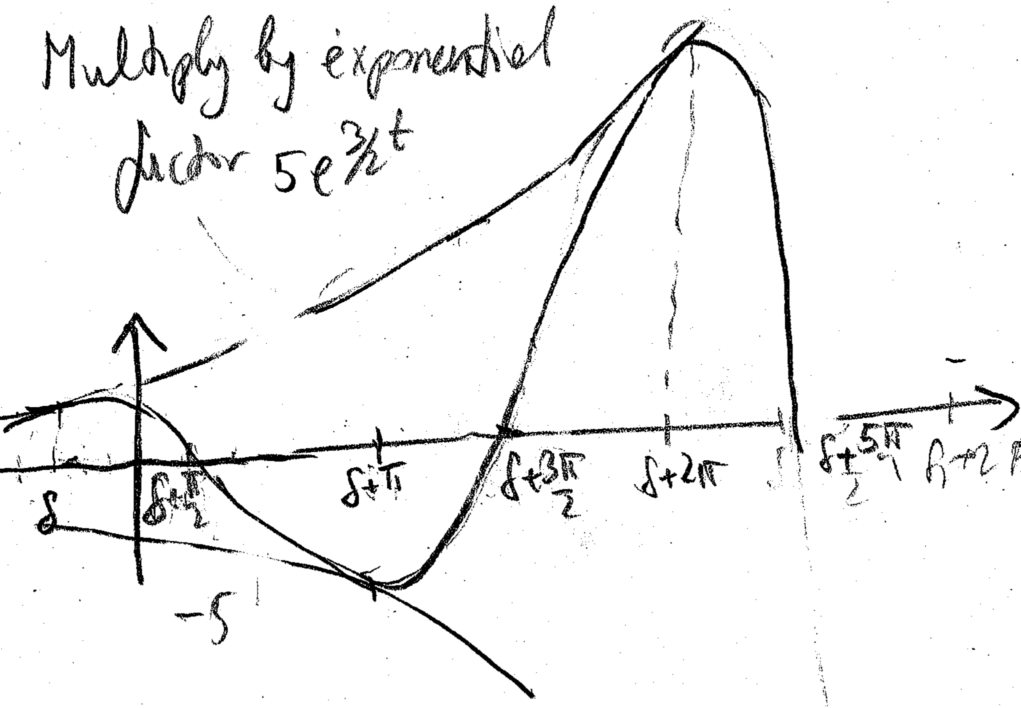
shifted by $\arctan\frac{4}{3}$ to the right

$\delta = -\arctan\frac{4}{3} \simeq -53.13°$

to the left



Multiply by exponential factor $5e^{3/2 t}$



Page 6 Problem 3 $y'' + (1-\alpha)y' + (2\alpha-3)(4-3\alpha)y = 0$

The characteristic equation

$$r^2 + (1-\alpha)r + (2\alpha-3)(4-3\alpha) = 0$$

By Vieta theorem $r_1 r_2 = (2\alpha-3)(4-3\alpha)$

$r_1 + r_2 = 1-\alpha$

$\Rightarrow$ we can take $r_1 = 2\alpha-3$

$r_2 = 4-3\alpha$

(using quadratic formula

$$D = (1-\alpha)^2 - 4(2\alpha-3)(4-3\alpha) =$$

$$= \alpha^2 - 2\alpha + 1 + 4\underbrace{(2\alpha-3)(3\alpha-4)}_{6\alpha^2 - 17\alpha + 12} =$$

$$= 25\alpha^2 - 70\alpha + 49 = (5\alpha-7)^2 \Rightarrow$$

$$r_1 = \frac{1-\alpha+5\alpha-7}{2} = 2\alpha-3,$$

$$r_2 = \frac{1-\alpha-5\alpha+7}{2} = 4-3\alpha \text{)}$$

(a) All solutions tend to 0 as $t \to +\infty$

$\Leftrightarrow$ $r_1 < 0$ and $r_2 < 0$

Indeed if $r_1 < 0$ and $r_2 < 0$ then

$$C_1 e^{r_1 t} + C_2 e^{r_2 t} \xrightarrow[t \to +\infty]{} 0 \quad \text{as } r_1 \neq r_2$$

and $(c_1+c_2t)e^{r_1t} \xrightarrow[t\to+\infty]{} 0$ as $r_1=r_2$

In the opposite direction, if any solution $y(t) \xrightarrow[t\to+\infty]{} 0$ then both r_1 & r_2 are negative.
Otherwise, if WLOG $r_1>0$, then e^{r_1t} is a solution and it does not tend to 0 as $t\to+\infty \Rightarrow$ contradiction

So $r_1<0$ and $r_2<0 \Leftrightarrow \begin{cases} 2\alpha-3<0 \\ 4-3\alpha<0 \end{cases} \Leftrightarrow$

$\begin{cases} \alpha<\frac{3}{2} \\ \alpha>\frac{4}{3} \end{cases} \Rightarrow \boxed{\frac{4}{3}<\alpha<\frac{3}{2}}$

(b) All nonzero solutions become unbounded if $r_1>0$ and $r_2>0$

Indeed if $r_1>0$ and $r_2>0$ then
$c_1e^{r_1t}+c_2e^{r_2t}$ (as $r_1\neq r_2$) or
$(c_1+c_2t)e^{r_1t}$ $(r_1=r_2)$ are unbounded as $c_1^2+c_2^2>0$)

In the opposite direction, if all non-zero solutions are unbounded, then both r_1 & r_2 are positive, otherwise, if WLOG $r_1<0$, then e^{r_1t} is bounded as $t\to+\infty$.

So $r_1 > 0$ and $r_2 > 0 \iff \begin{cases} 2\alpha - 3 > 0 \\ 4 - 3\alpha > 0 \end{cases} \iff \begin{cases} \alpha > \frac{3}{2} \\ \alpha < \frac{4}{3} \end{cases} \Rightarrow$

impossible to hold simultaneously $\Rightarrow$ $\boxed{\text{no such } \alpha}$