Homework # 4 Solutions, MATH308 - Summer 2012

Problem 1

Let $Q(t)$ is the amount of salt in tank at time t. Then $Q(0)=0$ and for $0 \le t \le 20$ min

$$Q'(t) = \frac{1}{6}\cdot 12 - \frac{Q(t)}{240}\cdot 12 = 2 - \frac{Q(t)}{20} \Rightarrow$$

$$Q(t) = 40 + C_1 e^{-\frac{t}{20}}$$

(recall that $Q' = aQ + b \Rightarrow$
$Q(t) = -\frac{b}{a} + Ce^{at}$)

Since $Q(0)=0 \Rightarrow C_1 = -40 \Rightarrow Q(t) = 40\left(1 - e^{-\frac{t}{20}}\right) \Rightarrow$

$$Q(20) = 40\left(1 - e^{-1}\right)$$

Further for $20 \le t \le 30$

$$Q'(t) = \frac{1}{2}\cdot 6 - \frac{Q(t)}{240}\cdot 6 = 3 - \frac{Q(t)}{40} \Rightarrow$$

$$Q(t) = 120 + C_2 e^{-\frac{t}{40}}$$

Since $Q(20) = 40(1-e^{-1}) \Rightarrow$

$$40(1-e^{-1}) = 120 + C_2 e^{-\frac{1}{2}} \Rightarrow$$

$$40 - 40e^{-1} = 120 + C_2 e^{-\frac{1}{2}} \Rightarrow$$

$$e^{-1/2} C_2 = -80 - 40e^{-1} \Rightarrow C_2 = -80e^{1/2} - 40e^{-1/2}$$

$\Downarrow$

For $20 \le t \le 30$

$$Q(t) = 120 - \left(80e^{1/2} + 40e^{-1/2}\right)e^{-t/40} \Rightarrow$$

$$Q(30) = 120 - \left(80e^{1/2} + 40e^{-1/2}\right)e^{-3/4} =$$

$$= \boxed{120 - \left(80e^{-1/4} + 40e^{-5/4}\right) = 40\left(3 - 2e^{-1/4} - e^{-5/4}\right)}$$

Problem 2

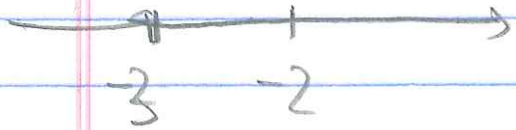
$$\left(t^2+5t+6\right)y' + (t-1)y = \sin t \Leftrightarrow$$

$$y' + \frac{t-1}{t^2+5t+6}y = \frac{\sin t}{t^2+5t+6}$$

$$P(t) = \frac{t-1}{t^2+5t+6} = \frac{t-1}{(t+2)(t+3)} \Rightarrow P(t) \text{ is discontinuous}$$

at $t=-3$ and $t=-2$

$$g(t) = \frac{\sin t}{(t+2)(t+3)} \Rightarrow \text{discontinuous at the same points as } P(t)$$



a) $t_0 = -4 < -3 \Rightarrow$ the solution of the IVP is certain to exist for $\boxed{t < -3}$

b) $t_0 = -5/2 \Rightarrow -3 < t_0 < -2 \Rightarrow$ the solution of the IVP is certain to exist for $\boxed{-2 < t < -3}$

c) $t_0 = -1 \Rightarrow t_0 > -2 \Rightarrow$ the solution of the IVP is certain to exist for $\boxed{t > -2}$