Homework # 5 MATH 308 Solutions

Problem 1

$$y'' + (\alpha+1)y' + (\alpha-2)(1-2\alpha)y = 0$$

The characteristic equation is

$$r^2 + (\alpha+1)r + (\alpha-2)(1-2\alpha) = 0 \qquad (*)$$

i) **Way 1** Finding roots using Vieta theorem

If r_1, r_2 are roots of $(*)$ then according to Vieta theorem

$$\begin{cases} r_1 + r_2 = -\alpha - 1 \\ r_1 r_2 = (\alpha-2)(1-2\alpha) \end{cases} \Rightarrow \text{easy to see that up to a swapping the roots } r_1 = \alpha-2,\ r_2 = 1-2\alpha$$

ii) **Way 2** Finding roots using quadratic formula

$$D = (\alpha+1)^2 - 4(\alpha-2)(1-2\alpha) = \alpha^2 + 2\alpha + 1 - 4(-2\alpha^2 + 5\alpha - 2) =$$

$$= 9\alpha^2 - 18\alpha + 9 = 9(\alpha-1)^2$$

$$r_1 = \frac{-(\alpha+1) + 3(\alpha-1)}{2} = \frac{2\alpha-4}{2} = \alpha - 2$$

$$r_2 = \frac{-(\alpha+1) - 3(\alpha-1)}{2} = \frac{-4\alpha+2}{2} = -2\alpha+1$$

The general solution is $y(t) = C_1 e^{(\alpha-2)t} + C_2 e^{(-2\alpha+1)t}$

(a) All solutions tend to zero $\Leftrightarrow$ $\alpha - 2 < 0$ **and** $-2\alpha + 1 < 0$ $\Rightarrow$

$$\begin{cases} \alpha - 2 < 0 \Leftrightarrow \alpha < 2 \\ -2\alpha + 1 < 0 \Leftrightarrow \alpha > \frac{1}{2} \end{cases} \Leftrightarrow \boxed{\frac{1}{2} < \alpha < 2}$$

(b) All nonzero solutions are unbounded $\Leftrightarrow$ $\alpha - 2 > 0$ and $-2\alpha + 1 > 0$

$\alpha - 2 > 0 \Leftrightarrow \alpha > 2$

$-2\alpha + 1 > 0 \Leftrightarrow \alpha < \frac{1}{2}$

No α can be simultaneously greater than 2 and less than $\frac{1}{2}$ $\Rightarrow$ $\boxed{\text{there is no such } \alpha}$

Problem 2 $\quad 16y''+24y'+9y=0 \qquad y(0)=-1,\ y'(0)=\alpha$

(a) The characteristic equation is

$$16r^2+24r+9=0 \iff (4r+3)^2=0 \Rightarrow r_1=r_2=-\frac{3}{4}$$

Alternatively, $D=24^2-\underbrace{4\cdot 9\cdot 16}_{2^6\cdot 3^2=24^2}=0$

$$r_1=r_2=\frac{-24}{32}=-\frac{3}{4}$$

$\Rightarrow$ The general solution is

$$y(t)=(C_1+C_2t)e^{-\frac{3}{4}t}=C_1e^{-\frac{3}{4}t}+C_2te^{-\frac{3}{4}t} \Rightarrow$$

$$y'(t)=-\frac{3}{4}C_1e^{-\frac{3}{4}t}+C_2e^{-\frac{3}{4}t}-\frac{3}{4}C_2te^{-\frac{3}{4}t}$$

$y(0)=-1 \Rightarrow C_1=-1$

$y'(0)=\alpha \Rightarrow -\frac{3}{4}\underbrace{C_1}_{-1}+C_2=\frac{3}{4}+C_2=\alpha \Rightarrow C_2=\alpha-\frac{3}{4} \Rightarrow$

$$y(t)=\left(-1+\left(\alpha-\frac{3}{4}\right)t\right)e^{-\frac{3}{4}t}$$

(b) $e^{(-)\frac{3}{4}t}\xrightarrow[t\to-\infty]{}+\infty$

i) $-1+\left(\alpha-\frac{3}{4}\right)t\xrightarrow[t\to-\infty]{}+\infty$ if $\alpha-\frac{3}{4}<0 \iff \alpha<\frac{3}{4} \Rightarrow y(t)\xrightarrow[t\to-\infty]{}+\infty$

ii) $-1+\left(\alpha-\frac{3}{4}\right)t\xrightarrow[t\to-\infty]{}-\infty$ if $\alpha-\frac{3}{4}>0 \iff \alpha>\frac{3}{4} \Rightarrow y(t)\xrightarrow[t\to-\infty]{}-\infty$

iii) Finally, if $\alpha-\frac{3}{4}=0$ then $-1+(\alpha-\frac{3}{4})t=-1\xrightarrow[t\to-\infty]{}-1 \Rightarrow y(t)\xrightarrow[t\to-\infty]{}-\infty$

Combining all this we conclude that

$$y(t)\xrightarrow[t\to-\infty]{}+\infty \iff \alpha<\frac{3}{4}$$

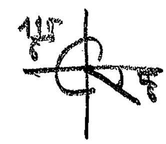
$$y(t)\xrightarrow[t\to-\infty]{}-\infty \iff \alpha\geq\frac{3}{4}$$

Problem 3

(a) i) $(3-2i)(10+5i) = 30-20i+15i+10 = \boxed{40-5i}$

ii) $$\frac{3-4i}{2-5i} = \frac{(3-4i)(2+5i)}{\underbrace{(2-5i)(2+5i)}_{2^2-(5i)^2 = 2^2+5^2}} = \frac{6-8i+15i+20}{4+25} = \frac{26}{29}+\frac{7}{29}i$$

(b) i) $$e^{-2-\frac{11\pi}{6}i} = e^{-2}\left(\cos\left(-\frac{11}{6}\pi\right)+i\sin\left(-\frac{11}{6}\pi\right)\right) =$$



$$= e^{-2}\left(\cos\left(\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{6}\right)\right) = e^{-2}\left(\underbrace{\cos\frac{\pi}{6}}_{\frac{\sqrt{3}}{2}}+i\underbrace{\sin\frac{\pi}{6}}_{\frac{1}{2}}\right) =$$

since $-\frac{11}{6}\pi = \frac{\pi}{6}-2\pi$

$$= \boxed{e^{-2}\frac{\sqrt{3}}{2}+i\frac{e^{-2}}{2}}$$

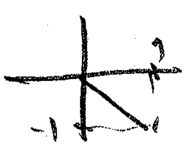
ii) $(1-i)^9$

Represent $(1-i)$ as $re^{i\theta}$

$(1-i) = re^{i\theta} \Rightarrow r = \sqrt{1+1} = \sqrt{2}$

$\cos\theta = \frac{1}{\sqrt{2}}$ $\Rightarrow$ we can take $\theta = -\frac{\pi}{4}$

$\sin\theta = -\frac{1}{\sqrt{2}}$



$\Rightarrow (1-i) = \sqrt{2}e^{-i\frac{\pi}{4}} \Rightarrow$

$$(1-i)^9 = \left(\sqrt{2}e^{-i\frac{\pi}{4}}\right)^9 = (\sqrt{2})^9 e^{-i\frac{9\pi}{4}} = \underbrace{2^{9/2}}_{2^4 2^{1/2}} e^{-i\frac{\pi}{4}} =$$

$-\frac{9\pi}{4} = -2\pi-\frac{\pi}{4}$

$$= 16\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right) = 16\sqrt{2}\left(\frac{1}{\sqrt{2}}-i\frac{1}{\sqrt{2}}\right) = \boxed{16(1-i)}$$

=1- **Problem 4** $\quad 9y''+12y'+20y=0$

1a) Characteristic equation

$$9r^2+12r+20=0$$

$$D=144-4\cdot 9\cdot 20=144-720=-576=i^2 24^2$$

$$r_{1,2}=\frac{-12\pm 24i}{18}=-\frac{2}{3}\pm\frac{4}{3}i$$

Alternatively, by completing square

$$9r^2+12r+20=9\left(r^2+\frac{4}{3}r+\frac{20}{9}\right)=$$

$$=9\left(\underbrace{r^2+2\cdot\frac{2}{3}r+\frac{4}{9}}_{\left(r+\frac{2}{3}\right)^2}+\frac{16}{9}\right)=9\left(\left(r+\frac{2}{3}\right)^2+\frac{16}{9}\right)=0\Rightarrow$$

$$\left(r+\frac{2}{3}\right)^2=-\frac{16}{9}\Leftrightarrow r+\frac{2}{3}=\pm\frac{4}{3}i\Rightarrow$$

$$r_{1,2}=-\frac{2}{3}\pm\frac{4}{3}i$$

$\Downarrow$

The general solution is

$$y(t)=e^{-\frac{2}{3}t}\left(C_1\cos\frac{4}{3}t+C_2\sin\frac{4}{3}t\right)$$

$$y\left(-\frac{\pi}{2}\right)=-3\Leftrightarrow e^{\frac{\pi}{3}}\left(C_1\cos\left(-\frac{2\pi}{3}\right)+C_2\sin\left(-\frac{2\pi}{3}\right)\right)=$$

$$=e^{\frac{\pi}{3}}\left(C_1\underbrace{\cos\frac{2\pi}{3}}_{-\frac{1}{2}}-C_2\underbrace{\sin\frac{2\pi}{3}}_{\frac{\sqrt{3}}{2}}\right)=e^{\frac{\pi}{3}}\left(-\frac{1}{2}C_1-\frac{\sqrt{3}}{2}C_2\right)=-3\Rightarrow$$

$$\frac{1}{2}C_1+\frac{\sqrt{3}}{2}C_2=3e^{-\frac{\pi}{3}}\Leftrightarrow C_1+\sqrt{3}C_2=6e^{-\frac{\pi}{3}}$$

$$y'(t)=-\frac{2}{3}e^{-\frac{2}{3}t}\left(C_1\cos\frac{4}{3}t+C_2\sin\frac{4}{3}t\right)+e^{-\frac{2}{3}t}\left(-\frac{4}{3}C_1\sin\frac{4}{3}t+\right.$$

$$\left.+\frac{4}{3}C_2\cos\frac{4}{3}t\right)\Rightarrow$$

$$y'\left(-\frac{\pi}{2}\right) = -\frac{2}{3}e^{\frac{\pi}{3}}\Big(C_1\underbrace{\cos\left(-\frac{2\pi}{3}\right)}_{-\frac{1}{2}} + C_2\underbrace{\sin\left(-\frac{2\pi}{3}\right)}_{-\frac{\sqrt{3}}{2}}\Big) + e^{\frac{\pi}{3}}\cdot$$

$$\Big(-\frac{4}{3}C_1\underbrace{\sin\left(-\frac{2\pi}{3}\right)}_{-\frac{\sqrt{3}}{2}} + \frac{4}{3}C_2\underbrace{\cos\left(-\frac{2\pi}{3}\right)}_{-\frac{1}{2}}\Big) =$$

$$= e^{\frac{\pi}{3}}\Big(\underline{\left(-\frac{2}{3}\right)\cdot\left(-\frac{1}{2}\right)C_1} + \left(-\frac{2}{3}\right)\cdot\left(-\frac{\sqrt{3}}{2}\right)C_2 + \underline{\left(-\frac{4}{3}\right)\cdot\left(-\frac{\sqrt{3}}{2}\right)C_1} +$$

$$+\frac{4}{3}\cdot\left(-\frac{1}{2}\right)C_2\Big) = e^{\frac{\pi}{3}}\left(\left(\frac{1}{3}+\frac{2\sqrt{3}}{3}\right)C_1 + \left(-\frac{2}{3}+\frac{\sqrt{3}}{3}\right)C_2\right) = 4$$

$$\Rightarrow \left(\frac{1}{3}+\frac{2\sqrt{3}}{3}\right)C_1 + \left(-\frac{2}{3}+\frac{\sqrt{3}}{3}\right)C_2 = 4e^{-\frac{\pi}{3}} \Leftrightarrow$$

$$(1+2\sqrt{3})C_1 + (-2+\sqrt{3})C_2 = 12e^{-\frac{\pi}{3}}$$

So, we have a system of 2 equations for C_1 & C_2

$$\begin{cases} C_1 + \sqrt{3}C_2 = 6e^{-\pi/3} & (\text{Eq } 1) \\ (1+2\sqrt{3})C_1 + (-2+\sqrt{3})C_2 = 12e^{-\frac{\pi}{3}} & (\text{Ep } 2) \end{cases}$$

To eliminate C_1:

Ep 2 $-(1+2\sqrt{3})$ Ep 1 :

$$\left(-2+\sqrt{3}-(1+2\sqrt{3})\sqrt{3}\right)C_2 = \left(-2+\cancel{\sqrt{3}}-\cancel{\sqrt{3}}-6\right)C_2 = -8C_2 =$$

$$= \left(12-(1+2\sqrt{3})\cdot 6\right)e^{-\pi/3} = (6-12\sqrt{3})e^{-\pi/3} \Rightarrow$$

$$C_2 = -\frac{1}{8}(6-12\sqrt{3})e^{-\pi/3} = \boxed{\left(\frac{3}{2}\sqrt{3}-\frac{3}{4}\right)e^{-\pi/3}}$$

From Eq 1

$$C_1 = 6e^{-\pi/3} - \sqrt{3}C_2 = \left(6-\sqrt{3}\left(\frac{3}{2}\sqrt{3}-\frac{3}{4}\right)\right)e^{-\pi/3} = \left(6-\frac{9}{2}+\frac{3\sqrt{3}}{4}\right)e^{-\pi/3}$$

$$= \boxed{\left(\frac{3}{2}+\frac{3\sqrt{3}}{4}\right)e^{-\pi/3}}$$

Therefore, $y(t) = \boxed{e^{-\frac{2}{3}t} e^{-\pi/3}\left(\left(\frac{3}{2}+\frac{3\sqrt{3}}{4}\right)\cos\frac{4}{3}t + \left(\frac{3\sqrt{3}}{2}-\frac{3}{4}\right)\sin\frac{4}{3}t\right)}$

$$= \boxed{e^{-\frac{2}{3}t-\frac{\pi}{3}} \frac{3}{4}\left((2+\sqrt{3})\cos\frac{4}{3}t + (2\sqrt{3}-1)\sin\frac{4}{3}t\right)}$$

$y(t) \to$ Since $\lim_{t\to+\infty} e^{-\frac{2}{3}t} = 0$ and $\cos\frac{4}{3}t$ and $\sin\frac{4}{3}t$ are bounded, then $\boxed{y(t) \xrightarrow[t\to+\infty]{} 0}$

b) $R = \sqrt{c_1^2+c_2^2}$; where c_1 and c_2 as in the previous item

$$c_2 = e^{-\pi/3}\left(\frac{3}{2}\sqrt{3} - \frac{3}{4}\right) = e^{-\frac{\pi}{3}} \cdot \frac{3}{4}(2\sqrt{3}-1)$$

$$c_1 = e^{-\pi/3}\left(\frac{3}{2}+\frac{3\sqrt{3}}{4}\right) = e^{-\frac{\pi}{3}} \cdot \frac{3}{4}(2+\sqrt{3})$$

$\Downarrow$

$$R = \sqrt{c_1^2+c_2^2} = e^{-\frac{\pi}{3}} \cdot \frac{3}{4} \sqrt{(2+\sqrt{3})^2+(2\sqrt{3}-1)^2} = e^{-\frac{\pi}{3}} \frac{3}{4} \times$$

$$\sqrt{4+\cancel{4\sqrt{3}}+3+12-\cancel{4\sqrt{3}}+1} = \boxed{e^{-\frac{\pi}{3}} \cdot \frac{3}{4}\sqrt{20}} = e^{-\pi/3} \cdot \frac{3}{2}\sqrt{5}$$

Also $c_1>0, c_2>0 \Rightarrow \delta$ is in the first quadrant $\Rightarrow$

$$\delta = \arctan\frac{c_2}{c_1} = \boxed{\arctan\frac{2\sqrt{3}-1}{2+\sqrt{3}}} = \arctan\frac{(2\sqrt{3}-1)(2-\sqrt{3})}{4-3} =$$

$$= \arctan(4\sqrt{3}-2-6+\sqrt{3}) = \arctan(5\sqrt{3}-8) \approx \boxed{0.583 \text{ rad or } 33.43^\circ}$$

So $\boxed{y(t) = e^{-\frac{\pi}{3}} \cdot \frac{3}{2}\sqrt{20}\, e^{-\frac{2}{3}t} \cos\left(\frac{4}{3}t - \underbrace{0.583}_{\delta}\right)}$

-7- Preliminary calculations for sketch

$$y(t) = e^{-\frac{\pi}{3}} \frac{3}{4}\sqrt{26}\, e^{-\frac{2}{3}t} \cos\left(\frac{4}{3}t - \delta\right)$$

$$y(t) = 0 \iff \frac{4}{3}t - \delta = \frac{\pi}{2} + \pi k, \ k \in \mathbb{Z}$$

$$t = \frac{3}{4}\left(\delta + \frac{\pi}{2} + \pi k\right) = \frac{3}{4}\delta + \frac{3\pi}{8} + \frac{3\pi k}{4}$$

The graph of $y(t)$ touch the graph of

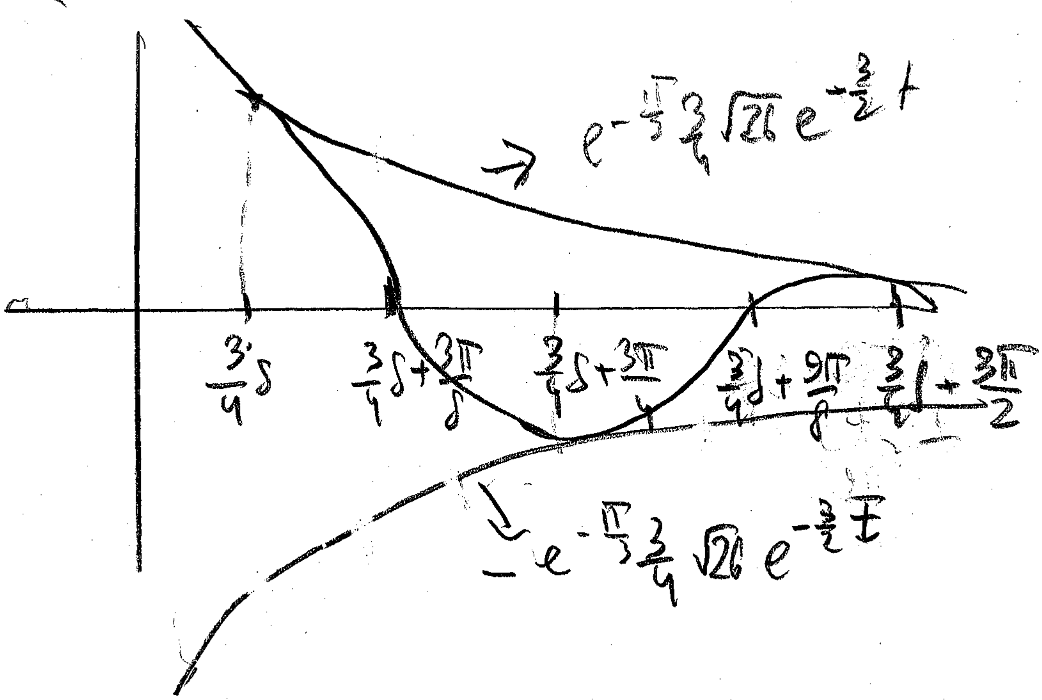
$$e^{-\frac{\pi}{3}} \frac{3}{4}\sqrt{26}\, e^{-\frac{2}{3}t} \iff \cos\left(\frac{4}{3}t - \delta\right) = \pm 1 \iff$$

$$\frac{4}{3}t - \delta = \pi k \iff t = \frac{3}{4}\left(\delta + \pi k\right) = \frac{3}{4}\delta + \frac{3\pi k}{4}$$

$k \in \mathbb{Z}$

The quasi-frequency $\mu = \frac{4}{3}$, the quasi-period is $T = \frac{2\pi}{\mu} = \frac{2\pi}{\frac{4}{3}} = \frac{3\pi}{2}$

Sketch



-8- Problem 5

a) $W = 4\,lb$,, $L = 24\,in = \frac{24}{12}\,ft = 2\,ft$

$k = \frac{W}{L} = \frac{4}{2} = 2\ lb/ft$

$\boxed{\frac{k}{m} = \frac{2}{\frac{1}{8}} = 16}$

$m = \frac{W}{g} = \frac{4}{32} = \frac{1}{8}$

(note that $\frac{k}{m} = \frac{\frac{W}{L}}{\frac{W}{g}} = \frac{g}{L}$ i.e. it is independent of W)

$\Downarrow$

Natural frequency

$\omega_0 = \sqrt{\frac{k}{m}} = \boxed{4}$

Period is

$T = \frac{2\pi}{\omega_0} = \boxed{\frac{\pi}{2}}$

$$\Rightarrow \begin{cases} \frac{1}{8}u'' + 2u = 0 \iff u'' + 16u = 0 \\ u(0) = 2\,in = \frac{1}{6}\,ft \\ u'(0) = -3\,in/s = -\frac{1}{4}\,ft/s \end{cases}$$

because upward

Gen. solution is

$u(t) = C_1 \cos 4t + C_2 \sin 4t \Rightarrow$

$u'(t) = -4C_1 \sin 4t + 4C_2 \cos 4t$

$u(0) = \frac{1}{6} \Rightarrow C_1 = \frac{1}{6}$

$u'(0) = -\frac{1}{4} \Rightarrow 4C_2 = -\frac{1}{4} \Rightarrow C_2 = -\frac{1}{16} \Rightarrow$

$u(t) = \frac{1}{6}\cos 4t - \frac{1}{16}\sin 4t$

$R = \sqrt{\left(\frac{1}{6}\right)^2 + \left(-\frac{1}{16}\right)^2} = \sqrt{\frac{1}{36} + \frac{1}{256}} = \boxed{\sqrt{\frac{73}{2304}}}$

$\cos\delta = \frac{C_1}{R} > 0$, $\sin\delta = \frac{C_2}{R} < 0 \Rightarrow \delta$ is in the forth quadrant

$\Rightarrow \delta = \arctan\frac{C_2}{C_1} = \arctan\frac{-\frac{1}{16}}{\frac{1}{6}} = \boxed{-\arctan\frac{3}{8}} \approx -0.359$ rad or $-20.556°$

Problem 5 b) $\gamma_{crit} = 2\sqrt{mk} = 2\sqrt{\frac{1}{8}\cdot 2} = 2\sqrt{\frac{1}{4}} = \boxed{1}$

$(mu'' + \gamma u' + ku = 0$

γ_{crit} is when $\gamma = \gamma_{crit}$ when $D = \gamma^2 - 4mk = 0$; i.e

$\gamma = 2\sqrt{mk}$ -