Homework # 5 MATH 308 Solutions

Problem 1

(a) $W(y_1, y_2)(t) = \begin{vmatrix} \cos(\ln t) & \sin(\ln t) \\ -\frac{\sin(\ln t)}{t} & \frac{\cos(\ln t)}{t} \end{vmatrix} =$

$$\frac{\cos^2(\ln t)}{t} + \frac{\sin^2(\ln t)}{t} = \frac{\cos^2(\ln t) + \sin^2(\ln t)}{t} = \frac{1}{t}$$

(b) Plug in $y_1(t) = \cos(\ln t)$

$y_1'(t) = -\sin(\ln t)\,\frac{1}{t}$

$y_1''(t) = -\cos(\ln t)\,\frac{1}{t^2} + \sin(\ln t)\,\frac{1}{t^2} = \frac{\sin(\ln t) - \cos(\ln t)}{t^2}$

$t^2 y_1'' + t y_1' + y_1 =$

$$= \cancel{t^2}\,\frac{\sin(\ln t) - \cos(\ln t)}{\cancel{t^2}} - \cancel{t}\,\frac{\sin(\ln t)}{\cancel{t}} + \cos(\ln t) =$$

$= 0$ ✓

$y_2(t) = \sin(\ln t)$

$y_2'(t) = \cos(\ln t)\,\frac{1}{t}$

$y_2''(t) = -\sin(\ln t)\,\frac{1}{t^2} - \cos(\ln t)\,\frac{1}{t^2}$

$t^2 y_2'' + t y_2' + y_2 = -\sin(\ln t) - \cos(\ln t) + \cos(\ln t) +$

$+ \sin(\ln t) = 0$ ✓

(c) By item 1(a) $W(y_1, y_2)(t) \neq 0 \Rightarrow$
y_1 and y_2 is a fundamental set of solutions $\Rightarrow$ gen. solution is
$$y(t) = C_1 \cos(\ln t) + C_2 \sin(\ln t)$$

(d) $y(0) = C_1 \cos 0 + C_2 \sin 0 = C_1 = 3$
$$y'(t) = -C_1 \sin(\ln t)\frac{1}{t} + C_2 \cos(\ln t)\frac{1}{t}$$
$y'(0) = C_2 = -5 \Rightarrow$

$$y(t) = 3\cos(\ln t) - 5\sin(\ln 2)$$

Problem 2. a) Characteristic equation is
$$r^2 + 2\alpha r + \alpha^2 - 9 = 0$$
$$D = 4\alpha^2 - 4\alpha^2 + 36 = 36$$
$$r_1 = \frac{-2\alpha + 6}{2} = 3 - \alpha \quad \Rightarrow$$
$$r_2 = \frac{-2\alpha - 6}{2} = -3 - \alpha$$
gen. solution is
$$y(t) = C_1 e^{(3-\alpha)t} + C_2 e^{-(\alpha+3)t}$$

All solutions tend to zero as $t \to +\infty \iff$

$3-\alpha<0$ and $-(\alpha+3)<0$

$$\begin{cases} 3-\alpha<0 \\ -(\alpha+3)<0 \end{cases} \iff \begin{cases} \alpha>3 \\ \alpha>-3 \end{cases} \iff \boxed{\alpha>3}$$

b) $\alpha=-1 \Rightarrow$ $r_1=3-\alpha=4$
$r_2=-3-\alpha=-2$

$\Rightarrow$ The general solution is

$$y(t)=C_1e^{4t}+C_2e^{-2t} \Rightarrow$$

$y(0)=C_1+C_2=-3$

$y'(t)=4C_1e^{4t}-2C_2e^{-2t} \Rightarrow y'(0)=4C_1-2C_2=2 \iff$

$2C_1-C_2=1 \Rightarrow$

$$\begin{cases} C_1+C_2=-3 \\ 2C_1-C_2=1 \end{cases} \Rightarrow 3C_1=-2 \Rightarrow C_1=-\frac{2}{3}$$

$C_2=-3-C_1=-3+\frac{2}{3}=-\frac{7}{3}$

$$y(t)=-\frac{2}{3}e^{4t}-\frac{7}{3}e^{-2t}$$

Problem 3 Our equation can be written as

$$y''-\frac{2}{t}y'+\frac{e^t}{t}y=0 \Rightarrow$$

By Abel's theorem $W(t):=W(y_1,y_2)(t)$ satisfies

$W' = \frac{2}{t} W \Rightarrow$

$W(t) = C\, e^{\int \frac{2}{t} dt} = C e^{2 \ln t} = C t^2$

If $W(5) = 2 \Rightarrow \quad 2 = C \cdot 25 \Rightarrow C = \frac{2}{25} \Rightarrow$

$W(3) = \frac{2}{25} \cdot 9 = \frac{18}{25}$