Homework assignment 5 solution, regular section

Problem 1

(a) $k = \frac{W}{L}$

$m = \frac{W}{g}$

$\Rightarrow \quad mu'' + ku = 0 \Leftrightarrow \quad \frac{W}{g} u'' + \frac{W}{L} u = 0 \Leftrightarrow$

$u'' + \frac{g}{L} u = 0$ (this shows that the value of W is actually not important for part a) of the problem)

$L = \frac{8}{3} \text{ in} = \frac{\cancel{8}^{2}}{3} \cdot \frac{1}{\cancel{12}_{3}} \text{ ft} = \frac{2}{9} \text{ ft} \Rightarrow$

$\frac{g}{L} = \frac{32}{2/9} = 16 \cdot 9 = 144 \Rightarrow \quad u'' + 144u = 0 \quad (1)$

$\Rightarrow \boxed{\omega_0 = \sqrt{144} = 12} \Rightarrow T = \frac{2\pi}{\omega_0} = \frac{2\pi}{12} = \boxed{\frac{\pi}{6}}$

Initial conditions: (IC) $\begin{cases} u(0) = -4 \text{ in} = -\frac{1}{3} \text{ ft} \\ u'(0) = 4 \text{ in/s} = 1/3 \text{ ft/s} \end{cases}$

(negative sign is because of pushing up)

(positive sign is because of downward velocity)

(1) $\Rightarrow$ char. equation is $r^2 + 144 = 0 \Leftrightarrow$

$r_{1,2} = \pm 12i \Rightarrow$ general solution is

$u(t) = C_1 \cos 12t + C_2 \sin 12t$

$u(0) = -\frac{1}{3} \Rightarrow \quad C_1 = -\frac{1}{3}$,

$u'(0) = 12 c_2 = \frac{1}{3} \Rightarrow c_2 = \frac{1}{36} \Rightarrow$

$$u(t) = -\frac{1}{3}\cos 12t + \frac{1}{36}\sin 12t$$

Amplitude

$R = \sqrt{c_1^2 + c_2^2} = \sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{1}{36}\right)^2} = \frac{1}{3}\sqrt{1+\left(\frac{1}{12}\right)^2}$

$= \frac{1}{3}\frac{\sqrt{145}}{12} = \boxed{\frac{\sqrt{145}}{36}}$

Phase

$\cos\delta = \frac{c_1}{R} = \frac{-1/3}{\frac{\sqrt{145}}{36}} = -\frac{12}{\sqrt{145}} < 0$

$\sin\delta = \frac{c_2}{R} = \frac{1/36}{\frac{\sqrt{145}}{36}} = \frac{1}{\sqrt{145}} > 0$

} δ is in the 2nd quadrant

$\Rightarrow \delta = \arccos\left(-\frac{12}{\sqrt{145}}\right) = -\arctan\frac{1}{12} + \pi \approx$

$\approx \boxed{3.06 \text{ rad} \approx 175.24^\circ}$

(b) $\gamma_{cnt} = 2\sqrt{mk}$ $\quad m = \frac{w}{g}, k = \frac{w}{L} \Rightarrow$

$\gamma_{cnt} = 2\sqrt{\frac{w^2}{Lg}} = \frac{2w}{Lg} = \frac{\cancel{2\cdot 8}}{\frac{2}{9}\cdot 32_2} = \boxed{\frac{9}{4}}$

Problem 2

(a) $28x^2y'' + 25xy' - 18y = 0, \; x > 0$

Indicial equation is

$$28r(r-1) + 25r - 18 = 0 \Leftrightarrow$$

$$28r^2 - 3r - 18 = 0$$

$$D = 9 + 2016 = 2025 = 45^2$$

$$r_1 = \frac{3+45}{56} = \frac{48}{56} = \frac{6}{7}$$

$$r_2 = \frac{3-45}{56} = -\frac{42}{56} = -\frac{3}{4}$$

$\Rightarrow$ gen. solution is

$$\boxed{y(x) = C_1 x^{6/7} + C_2 x^{3/4}}$$

(b) $4x^2y'' - 8xy' + 9y = 0, \; x > 0$

Indicial equation:

$$4r(r-1) - 8r + 9 = 0$$

$$4r^2 - 12r + 9 = 0$$

$$D = 144 - 4 \cdot 4 \cdot 9 = 0$$

$$r_{1,2} = \frac{12}{8} = \frac{3}{2} \Rightarrow$$

$\Rightarrow$ general solution is

$$\boxed{y(x) = (C_1 + C_2 \ln x)\, x^{3/2}}$$

(c) $x^2y'' - 13xy' + 113y = 0, \; x>0$

Indicial equation is

$$r(r-1) - 13r + 113 = 0$$

$$r^2 - 14r + 113 = 0$$

$$D = 196 - 4\cdot 113 = 196 - 452 = -256 = -16^2$$

$$r_{1,2} = \frac{14 \pm 16i}{2} = 7 \pm 8i$$

$\Rightarrow$ general solution is

$$\boxed{y(x) = x^7 \left(C_1 \cos(8 \ln x) + C_2 \sin(8 \ln x)\right)}$$

Problem 3

$$y'' - 4ty' + (4t^2-2)y = 0$$

$$y_1(t) = e^{t^2}$$

Look for a solution in the form

$$y(t) = v(t)e^{t^2} \Rightarrow$$

$$y'(t) = v(t)\cdot 2te^{t^2} + v'(t)e^{t^2}$$

$$y''(t) = v(t)\cdot(2e^{t^2} + 4t^2e^{t^2}) + 2v'(t)\cdot 2te^{t^2} + v''(t)e^{t^2}$$

$$\Rightarrow y'' - 4ty' + (4t^2-2)y = \underbrace{\left(\cancel{2+4t^2} - \cancel{8t^2} + \cancel{4t^2 - 2}\right)}_{0\text{, as expected}} e^{t^2}v$$

$$+ \left(\underbrace{4t - 4t}_{0\ \text{(this occurs occasionally)}}\right)e^{t^2}v' + e^{t^2}v'' = 0 \iff$$

$$e^{t^2}v'' = 0 \iff v'' = 0 \Rightarrow v' = C_1 \Rightarrow$$

$v = C_1t + C_2 \Rightarrow$ the general solution

$$\boxed{y(t) = (C_1t + C_2)e^{t^2}}$$

Problem 4

(a) $u'' - \arcsin(t^2)u' - 8\cos u = -t^4$

$x_1 = u$

$x_2 = u'$

$$\Rightarrow \boxed{\begin{cases} x_1' = x_2 \\ x_2'' = \arcsin(t^2)x_2 + 8\cos x_1 - t^4 \end{cases}}$$

(b) $y^{(3)} + 12\tan^2(t^3)y' - (2-4t)y = 0$

$x_1 = y$

$x_2 = y'$

$x_3 = y''$

$$\Rightarrow \begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = -12\tan^2(t^3)x_2 + (2-4t)x_1 \end{cases}$$

(a) $$\begin{cases} x_1' = 4x_1 + 8x_3 \\ x_2' = 20x_2 - 18x_3 \\ x_3' = 5x_1 - 13x_2 \end{cases} \Leftrightarrow \boxed{X' = \begin{pmatrix} 4 & 0 & 8 \\ 0 & 20 & -18 \\ 5 & -13 & 0 \end{pmatrix} X}$$

(b) $$\begin{cases} x' = (t^3 - 5t)x + \ln(3t^2+1)y - \cos(t^4) \\ y' = -(t^3 - 3t + 2)x + 6\sin(2t)y + 20 \end{cases} \Leftrightarrow$$

$$\boxed{\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} t^2 - 5t & \ln(3t^2+1) \\ -(t^2-3t+2) & 6\sin(2t) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -\cos(t^4) \\ 20 \end{pmatrix}}$$