Homework 5, Honors part solution

Problem 3B $ty'' - (1+3t)y' + 3y = 0$

$y_1 = e^{ct}$ for some c.

First, find c by plugging $y_1(t)$ into the equation:

$$\left(t c^2 - (1+3t)c + 3\right)e^{ct} = \left((c^2-3c)t + (3-c)\right)e^{ct} =$$

$$(c-3)(ct-1)e^{ct} = 0 \text{ for any } t \Rightarrow c-3=0 \Rightarrow c=3$$

Now look for a solution in the form $y(t) = v(t)e^{3t}$

$$\begin{array}{l|l}
3\times & y(t) = v(t)e^{3t} \\
-(1+3t)\times & y'(t) = 3v(t)e^{3t} + v'(t)e^{3t} \\
t\times & y''(t) = 9v(t)e^{3t} + 6v'(t)e^{3t} + v''(t)e^{3t}
\end{array}$$

$$ty'' - (1+3t)y' + 3y = (\underbrace{9t - 3(1+3t) + 3}_{0 \text{ (as expected)}})v + \left(6t - (1+3t)\right)v' +$$

$$+ te^{3t}v'' = 0 \Rightarrow$$

$$\left(tv'' + (3t-1)v'\right)e^{3t} = 0 \Rightarrow$$

$$tv'' + (3t-1)v' = 0$$

Let $w = v'$ $\Rightarrow$ $tw' + (3t-1)w = 0$ $\Rightarrow$ separate:

$$\frac{dw}{w} = \frac{1-3t}{t}dt = \left(\frac{1}{t} - 3\right)dt \Rightarrow \text{Integrand}$$

$$\int \frac{dw}{w} = \int\left(\frac{1}{t} - 3\right)dt \Rightarrow \ln|w| = \ln t - 3t + \tilde{C}_1 \Rightarrow$$

$$|w| = e^{\ln t - 3t + \tilde{C}_1} = e^{\tilde{C}_1}te^{-3t} \Rightarrow w = \underbrace{\pm e^{\tilde{C}_1}}_{C_1} te^{-3t}$$

So $w = \tilde{C}_1 t e^{-3t}$

Recall that $w = v' \Rightarrow v' = C_1 t e^{-3t} \Rightarrow$

$$v = \tilde{C}_1 \int t e^{-3t}\,dt + C_2 \underset{\substack{\text{integration} \\ \text{by parts}}}{=} \tilde{C}_1 \Big(-\frac{1}{3} t e^{-3t} +$$

$$+ \frac{1}{3} \int e^{-3t}\,dt \Big) + C_2 = \tilde{C}_1 \left(-\frac{1}{3} t e^{-3t} - \frac{1}{9} e^{-3t} \right) + C_2$$

$$= \underbrace{\left(-\frac{\tilde{C}_1}{9}\right)}_{C_1} (3t+1) e^{-3t} + C_2 = C_1 (3t+1) e^{-3t} + C_2$$

$\Downarrow$

$$y(t) = v(t) e^{3t} = \left(C_1 (3t+1) e^{-3t} + C_2 \right) e^{3t} =$$

$$= \boxed{C_1 (3t+1) + C_2 e^{3t}}$$