# Homework #5 Solutions, MATH 308 - Summer 2012

## Problem 1

$$2x\left(1+\sqrt{x^2-y}\right)dx-\sqrt{x^2-y}\;dy=0 \Rightarrow$$

$$P=2x\left(1+\sqrt{x^2-y}\right) \Rightarrow P_y=2x\cdot\left(-\frac{1}{2\sqrt{x^2-y}}\right)=-\frac{x}{\sqrt{x^2-y}}$$

$$Q=-\sqrt{x^2-y} \qquad Q_x=-\frac{2x}{2\sqrt{x^2-y}}=-\frac{x}{\sqrt{x^2-y}}$$

So $P_y=Q_x$ (on simply connected domain $\{(x,y):\ y<x^2\}$) $\Rightarrow$ the equation is exact.

Find the potential $\varphi$ of the field $\langle P,Q\rangle$

$$\begin{cases}\varphi_x=P\\ \varphi_y=Q\end{cases} \Leftrightarrow \begin{cases}\varphi_x=2x\left(1+\sqrt{x^2-y}\right)\\ \varphi_y=-\sqrt{x^2-y}\end{cases} \Rightarrow \varphi=-\int\sqrt{x^2-y}\,dy+h(x)=$$

$u=x^2-y \Rightarrow du=-dy$

$$=\frac{2}{3}(x^2-y)^{3/2}+h(x)$$

Substituting into the first equation, we get

$$\cancel{\frac{2}{3}\cdot\frac{3}{2}\,2x\sqrt{x^2-y}}+h'(x)=2x+\cancel{2x\sqrt{x^2-y}} \Rightarrow h'(x)=2x \Rightarrow$$

$h(x)=x^2+C$. So as a potential we can take

$$\varphi(x,y)=\frac{2}{3}(x^2-y)^{3/2}+x^2\Rightarrow$$

$\Rightarrow$ The general solution is

$$\boxed{\frac{2}{3}(x^2-y)^{3/2}+x^2=C}$$

**Problem 2**

$$(ax^2y^2+xy^3)+(x^3y+bx^2y^2)y'=0 \iff$$

$$\underbrace{(ax^2y^2+xy^3)}_{P}dx+\underbrace{(x^3y+bx^2y^2)}_{Q}dy=0$$

$$P_y = 2ax^2y+3xy^2 \Rightarrow$$

$$Q_x = 3x^2y+2bxy^2$$

The equation is exact $\iff P_y=Q_x \iff \begin{cases}2a=3\\ 3=2b\end{cases} \iff$

$$\boxed{a=\frac{3}{2},\ b=\frac{3}{2}}$$

Find a potential $\varphi$ of the vector field $\langle P,Q\rangle$ in ~~this case:~~

$$\varphi_x = \frac{3}{2}x^2y^2+xy^3 \Rightarrow \varphi=\int\left(\frac{3}{2}x^2y^2+xy^3\right)dx+h(y)=$$

$$\varphi_y = x^3y+\frac{3}{2}x^2y^2 \qquad =\frac{1}{2}x^3y^2+\frac{x^2}{2}y^3+h(y)$$

Substituting to the second equation:

$\cancel{x^3y^2+\frac{3}{2}x^2y^2}+h'(y) = \cancel{x^3y+\frac{3}{2}x^2y^2} \Rightarrow h'(y)=0 \Rightarrow$

$h(y) = \text{const} \Rightarrow$ as a potential we can take

$$\frac{1}{2}(x^3y^2+x^2y^3) \Rightarrow$$

The general solution is

$$\boxed{\frac{1}{2}(x^3y^2+x^2y^3) = C_1} \Leftrightarrow$$

$$x^3y^2+x^2y^3 = \underbrace{2C_1}_{C} \Leftrightarrow x^3y^2+x^2y^3=C$$

If $y(1)=2$ then $1^3\cdot 2^2 + 1^2\cdot 2^3 = C \Rightarrow C = 4+8=12 \Rightarrow$

The solution of IVP is $\boxed{x^3y^2+x^2y^3=12}$

<u>Problem 3</u> $\quad dx + \left(\frac{2x}{y}+\cos(y^3)\right)dy = 0$

$P=1$ $\qquad P_y=0$ $\qquad P_y \neq Q_x \Rightarrow$ the equation is not exact

$Q=\frac{2x}{y}+\cos(y^3)$ $\qquad Q_x=\frac{2}{y}$

Let us try to find an integrating factor

$$\underbrace{\mu\, dx}_{P} + \underbrace{\mu\left(\frac{2x}{y}+\cos(y^3)\right)}_{Q} dy$$

$$P_y = \mu_y \qquad Q_x = \mu_x\frac{2x}{y}+\frac{2\mu}{y}$$

Let us look for $\mu$ which <u>does not depend</u> on $x$

i.e. such that $\mu_x=0$

$\Rightarrow P_y = Q_x \Leftrightarrow \mu_y = \frac{2\mu}{y} \Rightarrow \frac{\mu_y}{\mu} = \frac{2}{y} \Leftrightarrow$
(separation)

$\underset{\text{(integration)}}{\Longrightarrow} \ln|\mu| = 2\ln|y| \Rightarrow \boxed{\mu = y^2} \Rightarrow$

$$y^2\left(dx + \left(\frac{2x}{y} + \cos(y^3)\right)dy\right) = 0 \quad \text{is exact}$$

$$y^2 dx + (2xy + y^2\cos y^3)\,dy$$

Let us find a function $\varphi$ such that

$$\begin{cases} \varphi_x = y^2 & \Rightarrow \varphi = \int y^2 dx + h(y) = xy^2 + h(y) \Rightarrow \\ \varphi_y = 2xy + y^2\cos y^3 & \end{cases}$$

$\Rightarrow$ (substituting to the second equation)

$$\cancel{2xy} + h'(y) = \cancel{2xy} + y^2\cos y^3 \Rightarrow$$

$$h(y) = \int y^2 \cos y^3\, dy = \frac{1}{3}\int \cos u\, du =$$

$$u = y^3 \Leftrightarrow du = 3y^2 dy$$

$$= \frac{1}{3}\sin u + C = \frac{1}{3}\sin y^3 + C \Rightarrow$$

We can take $\varphi = xy^2 + \frac{1}{3}\sin y^3 \Rightarrow$ the general solution is $\boxed{xy^2 + \frac{1}{3}\sin y^3 = C}$