Homework Assignment 6 in Differential Equations, MATH308-Soring 2017

due March 30, 2017

Topics covered: multiplication of matrices (section 7.2), determinant, eigenvalues and eigenvectors (section 7.3),fundamental set of solutions of system of first order (section 7.4), linear homogeneous systems with constant coefficients : the case of distinct real eigenvalues (section 7.5)

1. (a) Let $A = \begin{pmatrix} 3 & -8 & 4 \\ -1 & 5 & -6 \\ 7 & 5 & -6 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 3 & -2 \\ 4 & -5 & 1 \\ 4 & -2 & 3 \end{pmatrix}$. Compute $AB - 3BA$.

   (b) Let A be an upper triangular $n \times n$ matrix,

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ & a_{22} & a_{23} & \dots & a_{2n} \\ & & a_{33} & \dots & a_{3n} \\ & \text{0} & & \ddots & \vdots \\ & & & & a_{nn} \end{pmatrix}$$

   (in other words, all entries a_{ij} with $i > j$ are equal to zero). In all items below justify your answer:

   i. Calculate the determinant of A;

   ii. Find the eigenvalues of A;

   iii. Is the vector $\mathbf{v^1} = (1, 0, \dots, 0)^T$ an eigenvector of A? If yes, then to which eigenvalue does it corresponds?

   iv. Under what condition, for $i > 1$, the vector $\mathbf{v^i} = (\underbrace{0, \dots, 0}_{i-1 \text{ times}}, 1, 0 \dots, 0)^T$ is an eigenvector of A?

2. Given a system

$$\begin{cases} x_1' &= -6x_1 - 2x_2 - 5x_3 \\ x_2' &= -3x_1 + x_2 + x_3 \\ x_3' &= 12x_2 + 2x_2 + 11x_3 \end{cases} \qquad (1)$$

   and vector functions $\mathbf{x^1}(t) = \begin{pmatrix} e^{3t} \\ -2e^{3t} \\ -e^{3t} \end{pmatrix}$, $\mathbf{x^2}(t) = \begin{pmatrix} -e^{6t} \\ e^{6t} \\ 2e^{6t} \end{pmatrix}$, $\mathbf{x^3}(t) = \begin{pmatrix} -e^{-3t} \\ -e^{-3t} \\ e^{-3t} \end{pmatrix}$.

   (a) Prove that each vector function $\mathbf{x^1}(t)$, $\mathbf{x^2}(t)$. $\mathbf{x^3}(t)$ is a solution of system (1) (by substituting them into this system).

   (b) Does $\mathbf{x^1}(t)$, $\mathbf{x^2}(t)$, $\mathbf{x^3}(t)$ consititute a fundamental set of solution and what is the general solution of (1)? Justify you answer.

   (c) Based on the data given in the problem what are the eigenvalues and the corresponding eigenvectors of the matrix of the system (1)? (*Hint: you do not need to calculate here anything, just use the information given in the problem*).

3. Given the following system of linear differential equations:

$$\begin{cases} x_1' &= -16x_1 + 9x_2 \\ x_2' &= -30x_1 + 17x_2 \end{cases} \qquad (2)$$

   (a) Find the general solution of the system (2).

(b) Find the solution of the system (2) satisfying the initial conditions: $x_1(0) = -7, \quad x_2(0) = -11$.

(c) Find all α_1 and α_2 such that if $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ is the solution of of the system (2) with initial condition $x(0) = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$ then $x(t) \to 0$ as $t \to \infty$.

(d) Find all β_1 and β_2 such that if $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ is the solution of of the system (2) with initial condition $x(0) = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$ then $x(t) \to 0$ as $t \to -\infty$.

4. Consider the following system:

$$\begin{cases} x_1' &= 7x_1 + 11x_2 + 3x_3 \\ x_2' &= -4x_1 - 8x_2 - 3x_3 \\ x_3' &= 4x_1 + 10x_2 + 5x_3 \end{cases} \tag{3}$$

(a) Find the general solution of the system (3).

(b) Find the solution of the the system (3) satisfying the initial condition $\begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}$