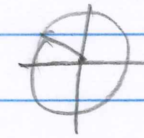
Homework #6 MATH 308 Solutions, Fall 2012

Problem 1

(a) $(2-3i)(3+5i) = 6-9i+10i+15 = \boxed{21+i}$

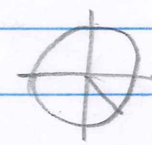
(b) $\dfrac{1+2i}{2+i} = \dfrac{(1+2i)(2-i)}{(2+i)(2-i)} = \dfrac{2+4i-i+2}{5} = \dfrac{4+3i}{5} = \boxed{\frac{4}{5}+\frac{3}{5}i}$

(c) $e^{\frac{5\pi}{6}i} = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6} = \boxed{-\frac{\sqrt{3}}{2}+\frac{i}{2}}$



(d) $e^{\left(-2-\frac{7\pi}{4}i\right)} = e^{-2}\left(\cos\left(-\frac{7\pi}{4}\right) + i\sin\left(-\frac{7\pi}{4}\right)\right) =$

$= e^{-2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right) = e^{-2}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = \frac{e^{-2}}{\sqrt{2}} - i\frac{e^{-2}}{\sqrt{2}}$



Problem 2

(a) Char. equation is

$5r^2+8r+5=0$

$D = 64-4\cdot 25 = -36$

$r = \dfrac{-8\pm 6i}{10} = -\frac{4}{5} \pm \frac{3}{5}i \Rightarrow \lambda = -\frac{4}{5},\ \mu = \frac{3}{5} \Rightarrow$

The gen. solution is

$\boxed{y(t) = c_1 e^{-\frac{4}{5}t}\cos\frac{3}{5}t + c_2 e^{-\frac{4}{5}t}\sin\frac{3}{5}t}$

(b) $y\left(\frac{5\pi}{6}\right) = c_1 e^{-\frac{4}{5}\cdot\frac{5}{6}\pi}\cos\underbrace{\left(\frac{3}{5}\cdot\frac{5}{6}\pi\right)}_{\frac{\pi}{2}} + c_2 e^{-\frac{4}{5}\cdot\frac{5}{6}\pi}\sin\underbrace{\left(\frac{3}{5}\cdot\frac{5}{6}\pi\right)}_{\frac{\pi}{2}} =$

$= c_2 e^{-\frac{2\pi}{3}} = -2 \Rightarrow \boxed{c_2 = -2e^{\frac{2\pi}{3}}}$

$$y'(t) = -\frac{4}{5}C_1 e^{-\frac{4}{5}t}\cos\frac{3}{5}t - \frac{3}{5}C_1 e^{-\frac{4}{5}t}\sin\frac{3}{5}t -$$

$$-\frac{4}{5}C_2 e^{-\frac{4}{5}t}\sin\frac{3}{5}t + \frac{3}{5}C_2 e^{-\frac{4}{5}t}\cos\frac{3}{5}t$$

$$y'\left(\frac{5\pi}{6}\right) \underset{\downarrow}{=} -\frac{3}{5}C_1 e^{-\frac{2}{3}\pi} - \frac{4}{5}C_2 e^{-\frac{2}{3}\pi} = -4 \Rightarrow$$

using that
$\cos\frac{\pi}{2}=0$, $\sin\frac{\pi}{2}=1$

$$3C_1 e^{-\frac{2}{3}\pi} + 4\left(-2e^{\frac{2\pi}{3}}\right)e^{-\frac{2\pi}{3}} = 20$$

$$3C_1 e^{-\frac{2}{3}\pi} - 8 = 20 \Rightarrow 3C_1 e^{-\frac{2}{3}\pi} = 28 \Rightarrow$$

$$C_1 = \frac{28}{3}e^{\frac{2}{3}\pi} \Rightarrow$$

$$\boxed{y(t) = \frac{28}{3}e^{\frac{2}{3}\pi - \frac{4}{5}t}\cos\frac{3}{5}t - 2e^{\frac{2}{3}\pi - \frac{4}{5}t}\sin\frac{3}{5}t}$$

$$y(t) = \underset{\underset{0}{\downarrow}}{e^{-\frac{4}{5}t}}\, e^{\frac{2\pi}{3}}\underbrace{\left(\frac{28}{3}\cos\frac{3}{5}t - 2\sin\frac{3}{5}t\right)}_{\text{bounded}} \xrightarrow[t\to+\infty]{} \boxed{0}$$

Problem 3 a) Char. eq. is $4r^2 - 20r + 25 = 0$

$D = 400 - 4\cdot 4\cdot 25 = 0$

$r_1 = r_2 = \frac{20}{8} = \frac{5}{2}$ - repeated roots

(Actually $4r^2 - 20r + 25 = (2r-5)^2$) $\Rightarrow$

The general solution is

$$C_1 e^{\frac{5}{2}t} + C_2 t e^{\frac{5}{2}t} = (C_1 + C_2 t)e^{\frac{5}{2}t}$$

(b) $y(0) = C_1 = \alpha$

$y'(t) = \frac{5}{2}C_1 e^{\frac{5}{2}t} + C_2 e^{\frac{5}{2}t} + \frac{5}{2}C_2 t e^{\frac{5}{2}t} \Rightarrow$

$y'(0) = \frac{5}{2}C_1 + C_2 = 3 \Rightarrow C_2 = 3 - \frac{5}{2}\alpha$

$\Downarrow$

$y(t) = \left(\alpha + \left(3 - \frac{5}{2}\alpha\right)t\right)e^{\frac{5}{2}t}$

If $3 - \frac{5}{2}\alpha > 0 \Leftrightarrow \alpha < \frac{6}{5}$ then $y(t) \xrightarrow[t \to +\infty]{} +\infty$

If $3 - \frac{5}{2}\alpha < 0 \Leftrightarrow \alpha > \frac{6}{5}$ then $y(t) \xrightarrow[t \to +\infty]{} -\infty$

If $3 - \frac{5}{2}\alpha = 0 \Leftrightarrow \alpha = \frac{6}{5}$ then $y(t) = \frac{6}{5}e^{\frac{5}{2}t} \to +\infty$

$\Downarrow$

$y(t) \xrightarrow[t \to +\infty]{} +\infty \Leftrightarrow \alpha \leq \frac{6}{5}$

$y(t) \xrightarrow[t \to +\infty]{} -\infty \Leftrightarrow \alpha > \frac{6}{5}$