Homework assignment 6 solutions MATH 308 - Summer 2012

Problem 1

a) $W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = \begin{vmatrix} e^t & t^2+2t+2 \\ e^t & 2t+2 \end{vmatrix} =$

$= (2t+2)e^t - (t^2+2t+2)e^t = \boxed{-t^2 e^t}$

b) Substitute $y_1(t)$ into equation (1): $(e^t)' = e^t, (e^t)'' = e^t$

$t e^t - (t+2)e^t + 2e^t = (t - t - 2 + 2)e^t = 0 \Rightarrow y_1(t)$ is a solution

Substitute $y_2(t)$ into equation (1): $y_2'(t) = 2t+2$, $y_2''(t) = 2$

$2t - (t+2)(2t+2) + 2(t^2+2t+2) = \cancel{2t} - \cancel{2t^2} - \cancel{4t} - \cancel{2t} - \cancel{4} +$
$+ \cancel{2t^2} + \cancel{4t} + \cancel{4} = 0 \Rightarrow y_2(t)$ is a solution

c) For this show that $W(y_1, y_2)(t) \neq 0$ for $t > 0$

Indeed by item a) $W(y_1, y_2)(t) = -t^2 e^t < 0$ for $t > 0 \Rightarrow W(y_1, y_2)(t) \neq 0$

d) By item c) the general solution is

$y(t) = C_1 e^t + C_2 (t^2+2t+2)$

Let us find C_1 and C_2 such that $y(1) = 0$, $y'(1) = 1$:

$$\begin{matrix} y(1) = 0 \Leftrightarrow \\ y'(1) = 1 \Leftrightarrow \end{matrix} \begin{cases} eC_1 + 5C_2 = 0 \\ eC_1 + 4C_2 = 1 \end{cases} \Rightarrow$$

Subtracting the second equation from the first one we get $\boxed{C_2 = -1} \Rightarrow eC_1 - 5 = 0 \Rightarrow$ $\boxed{C_1 = 5e^{-1}} \Rightarrow$

$$\boxed{y(t) = 5e^{t-1} - (t^2+2t+2)}$$

Problem 2 a) The characteristic equation is

$$r^2 - 3r - 4 = 0$$

$$D = 9 + 16 = 25$$

$$r_1 = \frac{3+\sqrt{25}}{2} = \frac{3+5}{2} = 4$$

$$r_2 = \frac{3-\sqrt{25}}{2} = \frac{3-5}{2} = -1$$

$r_1 \neq r_2$ and they are real $\Rightarrow$ e^{4t}, e^{-t} is a fundamental set of solutions $\Rightarrow$ gen. solution is

$$\boxed{y(t) = C_1 e^{4t} + C_2 e^{-t}}$$

b) $y(0) = \alpha \iff C_1 + C_2 = \alpha$

$y'(t) = 4C_1 e^{4t} - C_2 e^{-t} \Rightarrow$

$y'(0) = 1 \iff 4C_1 - C_2 = 1$

Therefore we get the system $\begin{cases} C_1 + C_2 = \alpha \\ 4C_1 - C_2 = 1 \end{cases}$

Eliminate C_2 by adding the equations: $5C_1 = \alpha + 1 \Rightarrow C_1 = \frac{\alpha+1}{5}$

From the first equation: $C_2 = \alpha - C_1 = \alpha - \frac{\alpha+1}{5} = \frac{5\alpha - \alpha - 1}{5} = \frac{4\alpha - 1}{5} \Rightarrow$

$$\boxed{y(t) = \frac{\alpha+1}{5} e^{4t} + \frac{4\alpha-1}{5} e^{-t}}$$

c) $y(t) \xrightarrow[t \to +\infty]{} 0 \iff$ the coefficients of e^{4t} is equal to zero, i.e. $\frac{\alpha+1}{5} = 0 \iff \boxed{\alpha = -1}$

d) $y(t) \xrightarrow[t \to -\infty]{} 0 \iff$ the coefficient of e^{-t} is equal to zero, i.e. $\frac{4\alpha - 1}{5} = 0 \iff \boxed{\alpha = \frac{1}{4}}$

Problem 3 By Abel's theorem the Wronskian W of a fundamental set of solutions satisfies the equation

$$W' + pW = 0$$

In our case $ty'' + 3y' + (\cos t)y = 0 \iff y'' + \frac{3}{t}y' + \frac{\cos t}{t}y = 0$

i.e $p = \frac{3}{t} \Rightarrow W' + \frac{3}{t}W = 0 \iff$ (after separation)

$$\frac{dW}{W} = -\frac{3\,dt}{t} \Rightarrow \text{(integrating)}$$

$$\ln|W| = -3\ln|t| + C_1 \Rightarrow$$

$$W(t) = C_1 t^{-3}$$

It is given that $W(3) = 1 \Rightarrow C_1 \cdot 3^{-3} = 1 \Rightarrow C_1 = 3^3 = 27 \Rightarrow$

$$W(4) = 27 \cdot 4^{-3} = \boxed{\frac{27}{64}}$$