 Homework #7 solutions MATH 308

Problem 1

a) Calculate the determinant of the matrix of columns:

$$\begin{vmatrix} -4e^{-5t} & e^{3t}\cos 2t & e^{3t}\sin 2t \\ 3e^{-5t} & -e^{3t}\sin 2t & e^{3t}\cos 2t \\ 2e^{-5t} & -2e^{3t}\sin 2t & 2e^{3t}\cos 2t \end{vmatrix} =$$

$$= e^{-5t}\left(-4e^{6t}\underbrace{\begin{vmatrix} -\sin 2t & \cos 2t \\ -2\sin 2t & 2\cos 2t \end{vmatrix}}_{0} - 3e^{6t}\underbrace{\begin{vmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{vmatrix}}_{2} + 2e^{6t}\underbrace{\begin{vmatrix} \cos 2t & \sin 2t \\ -\sin 2t & \cos 2t \end{vmatrix}}_{1}\right) = e^{-5t}e^{6t}(-6+2) = -4e^{t} \neq 0$$

expansion w.r.t. the first column

$\Rightarrow$ It is the fundamental set of solution and the general solution is

$$C_1\begin{pmatrix} -4e^{-5t} \\ 3e^{-5t} \\ 2e^{-5t} \end{pmatrix} + C_2\begin{pmatrix} e^{3t}\cos 2t \\ -e^{3t}\sin 2t \\ -2e^{3t}\sin 2t \end{pmatrix} + C_3\begin{pmatrix} e^{3t}\sin 2t \\ e^{3t}\cos 2t \\ 2e^{3t}\cos 2t \end{pmatrix}$$

<u>Remark</u> It was enough to calculate the determinant only at one time moment, for example for $t=0$

$$\begin{vmatrix} -4 & 1 & 0 \\ 3 & 0 & 1 \\ 2 & 0 & 2 \end{vmatrix} = -\begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} = -4 \neq 0 \Rightarrow$$

the same conclusion

(b) Calculate the determinant of matrix of columns

$$\begin{vmatrix} 10e^{-7t} & -5e^{-7t} \\ -14e^{-7t} & 7e^{-7t} \end{vmatrix} = e^{-14}\left(70-70\right)=0 \Rightarrow$$

this is not a fundamental set of solutions.

Problem 2 $\begin{cases} x_1' = -8x_1 - 10x_2 \\ x_2' = 5x_1 + 7x_2 \end{cases}$

1a) $A = \begin{pmatrix} -8 & -10 \\ 5 & 7 \end{pmatrix}$

1) The characteristic equation is $\lambda^2 - \operatorname{tr}A\,\lambda + \det A$

$\operatorname{tr} A = -8+7 = -1$

$\det A = -56+50 = -6$

The characteristic equation is

$\lambda^2 + \lambda - 6 = 0$

$D = 1 - 4\cdot(-6) = 1+24 = 25$

$\lambda_1 = \frac{-1+5}{2} = 2$, $\lambda_2 = \frac{-1-5}{2} = -3$

2) An eigenvector of $\lambda_1 = 2$

$(A-2I)v = \begin{pmatrix} -10 & -10 \\ 5 & 5 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow$ $v_1 + v_2 = 0$. Take $v_2 = 1 \Rightarrow v_1 = -1$

$\Rightarrow$ $v^1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is an eigenvector of A corresponding to $\lambda_1 = 2 \Rightarrow e^{2t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is a solution

Page 3

3) An eigenvector, corresponding to $\lambda_2 = -3$

$$(A-(-3)I)v = (A+3I)v = \begin{pmatrix} -5 & -10 \\ 5 & 10 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \Leftrightarrow$$

$5v_1 + 10v_2 = 0 \Leftrightarrow v_1 + 2v_2 = 0$. Take $v_2 = 1 \Rightarrow v_1 = -2$

$\Rightarrow v = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ is an eigenvector of $\lambda_2 = -3 \Rightarrow$

$e^{-3t}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ is a solution $\Rightarrow$

$$\boxed{x(t) = c_1 e^{2t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 e^{-3t}\begin{pmatrix} -2 \\ 1 \end{pmatrix}}$$ is the general solution

(b) $\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \Rightarrow$

$$c_1\begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{matrix} 3 \\ -2 \end{matrix}$$

$$\left(\begin{array}{cc|c} -1 & -2 & 3 \\ 1 & 1 & -2 \end{array}\right) \overset{R_2 \to R_2 + R_1}{\sim} \left(\begin{array}{cc|c} -1 & -2 & 3 \\ 0 & -1 & 1 \end{array}\right) \Rightarrow$$

$-c_1 - 2c_2 = 3 \quad -c_1 + 2 = 3 \Rightarrow c_1 = -1$

$-c_2 = 1 \Rightarrow c_2 = -1$

$$\Rightarrow \boxed{x(t) = -e^{2t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} - e^{-3t}\begin{pmatrix} -2 \\ 1 \end{pmatrix} = e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{-3t}\begin{pmatrix} 2 \\ -1 \end{pmatrix}}$$

(c) $x(t) = c_1 e^{2t}\begin{pmatrix}-1\\1\end{pmatrix} + c_2 e^{-3t}\begin{pmatrix}-2\\1\end{pmatrix} \xrightarrow[t\to+\infty]{} 0 \iff$

$c_1 = 0$, i.e. $x(t) = c_2 e^{-3t}\begin{pmatrix}-2\\1\end{pmatrix} \iff x(0) = c_2\begin{pmatrix}-2\\1\end{pmatrix}$

$\iff \begin{pmatrix}\alpha_1\\\alpha_2\end{pmatrix} = \begin{pmatrix}-2c_2\\c_2\end{pmatrix} \iff \boxed{\alpha_1 = -2\alpha_2}$

the Eigenline of $\lambda = -3$

(d) $x(t) = c_1 e^{2t}\begin{pmatrix}-1\\1\end{pmatrix} + c_2 e^{-3t}\begin{pmatrix}-2\\1\end{pmatrix} \xrightarrow[t\to-\infty]{} 0 \iff$

$c_2 = 0$ i.e. $x(t) = c_1 e^{2t}\begin{pmatrix}-1\\1\end{pmatrix} \iff x(0) = c_1\begin{pmatrix}-1\\1\end{pmatrix}$

$\iff \begin{pmatrix}\beta_1\\\beta_2\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix} \iff \boxed{\beta_1 = -\beta_2}$

the eigenline of $\lambda = 2$

Problem 3

$\det(A-\lambda I) = \begin{vmatrix}-2-\lambda & 3 & -1\\ 4 & -1-\lambda & 1\\ -4 & -3 & -5-\lambda\end{vmatrix} =$

$= -(\lambda+2)\left((\lambda+1)(\lambda+5)+3\right) - 3\left(-4(\lambda+5)+4\right) -$

$-\left(-12 - 4(\lambda+1)\right) = -(\lambda+2)(\lambda^2+6\lambda+8) -$

$-3(-4\lambda-16) + 4\lambda + 16 =$

$= -\lambda^3 - 2\lambda^2 - 6\lambda^2 \cancel{-12\lambda} - 8\lambda \cancel{-16} +$

$+ \cancel{12\lambda} + 48 + 4\lambda + \cancel{16} = -\lambda^3 - 8\lambda^2 - 4\lambda + 48 = 0$

$\lambda^3 + 8\lambda^2 + 4\lambda - 48 = 0$

Integer roots are divisors of 48 i.e. among

$\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 16; \pm 24$

Substitute to $f(\lambda) = \lambda^3 + 8\lambda^2 + 4\lambda - 48$

$f(1) = 1 + 8 + 4 - 48 = -35 \neq 0$ → not a root

$f(-1) = -1 + 8 - 4 - 48 = -45 \neq 0$ → not a root

$f(2) = 8 + 8 \cdot 4 + 8 - 48 = 48 - 48 = 0 \Rightarrow$

$\boxed{\lambda = 2}$ is a root $\Rightarrow \lambda^3 + 8\lambda^2 + 4\lambda - 48$ is divisible by $\lambda - 2$

$$
\begin{array}{r|l}
 & \lambda^2 + 10\lambda + 24 \\
\hline
\lambda - 2 & \lambda^3 + 8\lambda^2 + 4\lambda - 48 \\
 & \underline{\lambda^3 - 2\lambda^2} \\
 & 10\lambda^2 + 4\lambda \\
 & \underline{10\lambda^2 - 20\lambda} \\
 & 24\lambda - 48 \\
 & \underline{24\lambda - 48} \\
 & 0
\end{array}
$$

Now solve the quadratic equation

$\lambda^2 + 10\lambda + 24 = 0$

$D = 100 - 4 \cdot 24 = 4$

$$\lambda_1 = \frac{-10+2}{2} = -4$$

$$\lambda_2 = \frac{-10-2}{2} = -6$$

So the eigenvalues $\boxed{\lambda = 2, -4, -6}$