Homework #7 Solutions Fall 2012

Problem 1 $4t^2y''-4ty'-5y=0, \quad t>0$

$y_1(t)=t^{-1/2}$

Find the second independent solution using the method of reduction of order:

For this look for a solution in the form $y(t)=v(t)y_1(t)$

$$\begin{array}{l|l}
-5 & y(t)=v(t)y_1(t)=v(t)t^{-1/2} \Rightarrow \\
-4t \; \times & y'(t)= -\frac{1}{2}v(t)t^{-3/2}+v'(t)t^{-1/2} \\
4t^2 & y''(t)= \frac{3}{4}v(t)t^{-5/2}\underbrace{-\frac{1}{2}v'(t)t^{-3/2}-\frac{1}{2}v'(t)t^{-3/2}}_{-v'(t)t^{-3/2}}+v''(t)t^{-1/2}
\end{array}$$

$$4t^2y''(t)-4ty'(t)-5y(t)=\underbrace{(3+2-5)}_{0}t^{-1/2}v(t)+(\underbrace{-4-4}_{-8})t^{1/2}v'+$$

$$+4t^{3/2}v''=-4t^{1/2}(2v'-tv'')=0 \Rightarrow$$

$$tv''-2v'=0$$

Let $w=v' \Rightarrow \quad tw'-2w=0 \Leftrightarrow \frac{w'}{w}=\frac{2}{t} \Rightarrow$

$$\ln|w|=2\ln t+c_1$$

$$w=e^{2\ln t+c_1}=\underbrace{e^{c_1}}_{C_2}t^2=Ct^2$$

$$v=C_2\int t^2\,dt+C_3=C_2\frac{t^3}{3}+C_3=\underbrace{\frac{C_2}{3}}_{C}t^3+C_3=Ct^3+C_3$$

$$\Downarrow$$

$$y(t)=v(t)t^{-1/2}=Ct^{5/2}+C_3t^{-1/2} \Rightarrow$$

As a second independent solution one can take

$$y_2(t)=\boxed{t^{5/2}}$$

Problem 2

a) i) $3y'' + 5y' - 2y = 7e^{2t}$

Char. eq. is $3r^2+5r-2=0$

$D = 25+24 = 49$

$r_1 = \frac{-5+7}{6} = \frac{1}{3}$

$r_2 = \frac{-5-7}{6} = -2$

$\alpha = 2 \Rightarrow \alpha$ is not a root $\Rightarrow s=0 \Rightarrow$

$$\boxed{Y_p(t) = Ae^{2t}}$$

ii) $3y'' + 5y' - 2y = 7e^{-2t}$

The char. equation is the same as in the item i) $\Rightarrow r_1 = \frac{1}{3}, r_2 = -2$

But $\alpha = -2 \Rightarrow \alpha$ is a non-repeated root $\Rightarrow s=1 \Rightarrow$

$$\boxed{Y_p(t) = Ate^{-2t}}$$

iii) $3y'' + 5y' - 2y = 3e^{2t}\cos 10t$

The char. eq. is the same as in the item i) $\Rightarrow r_1 = \frac{1}{3}, r_2 = -2$

But $\alpha + i\beta = 2+10i \Rightarrow \alpha + i\beta$ is not a root $\Rightarrow s=0 \Rightarrow$

$$\boxed{Y_p(t) = A_1 e^{2t}\cos 10t + A_2 e^{2t}\sin 10t}$$

iv) $3y'' + 5y' - 2y = 7e^{-2t}\cos 3t$

The char. ep. is the same as in the item i) $\Rightarrow r_1 = \frac{1}{3}, r_2 = -2$

But $\alpha + i\beta = -2+10i \Rightarrow \alpha + i\beta$ is not a root $\Rightarrow s=0 \Rightarrow$

$$\boxed{Y_p(t) = A_1 e^{-2t}\cos 3t + A_2 e^{-2t}\sin 2t}$$

v) $4y'' - 4y' + y = 5e^{3t}$

The char. eq is $4r^2 - 4r + 1 = 0 \Leftrightarrow (2r-1)^2 = 0 \Rightarrow r_1 = r_2 = \frac{1}{2}$

$\alpha = 3$ is not a root of the char. eq. $\Rightarrow$

$$\boxed{Y_p(t) = Ae^{3t}}$$

(vi) $4y'' - 4y' + y = 5e^{t/2} - 2e^{t/2}\sin 3t$

The char. eq. is the same as in item (v) $\Rightarrow r_1 = r_2 = \frac{1}{2}$

Consider separately two equations

$$4y'' - 4y' + y = 5e^{t/2} \qquad (a)$$

$$4y'' - 4y' + y = -2e^{t/2}\sin 3t \qquad (b)$$

For (a) $\alpha = \frac{1}{2}$ is a repeated root $\Rightarrow s = 2 \Rightarrow$ so a particular solution for (a) can be found in the form $Y_p^1(t) = At^2e^{t/2}$

For (b) $\alpha + i\beta = \frac{1}{2} + 3i \Rightarrow \alpha + i\beta$ is not a root of the char. eq. $\Rightarrow s = 0 \Rightarrow$ a particular solution for (b) can be found in the form

$$Y_p^2(t) = A_1 e^{t/2}\cos 3t + A_2 e^{t/2}\sin 3t$$

$\Downarrow$

A particular solution of the original equation can be found as the sum of $Y_p^1(t)$ and $Y_p^2(t)$:

$$\boxed{Y_p(t) = At^2e^{t/2} + A_1 e^{t/2}\cos 3t + A_2 e^{t/2}\sin 3t}$$

(vii) $y'' + \omega_0^2 y = \cos\omega t + 2\sin\omega t$

Char. eq. is $r^2 + \omega_0^2 = 0 \Rightarrow r = \pm i\omega_0$

Case 1 $\omega^2 \neq \omega_0^2 \Rightarrow \alpha + i\beta = i\omega \Rightarrow \alpha + i\beta$ is not a root of char. eq. $\Rightarrow s = 0 \Rightarrow$

$$\boxed{Y_p(t) = A_1\cos\omega t + A_2\sin\omega t}$$

Case 2 $\omega^2 = \omega_0^2 \Rightarrow \alpha + i\beta = i\omega_0 \Rightarrow \alpha + i\beta$ is a root of char. eq $\Rightarrow s = 1 \Rightarrow \boxed{Y_p(t) = t(A_1\cos\omega_0 t + A_2\sin\omega_0 t)}$

(VIII) $18y'' + 30y' + 17y = e^{-5t/6}\left(\cos\frac{t}{4} - 2\sin\frac{t}{4}\right)$

Char. eq. is $18r^2 + 30r + 17 = 0$

$$D = 900 - \underbrace{4 \cdot 18 \cdot 17}_{1224} = -324 = -18^2$$

$$r_{1,2} = \frac{-30 \pm 18i}{36} = -\frac{5}{6} \pm \frac{i}{2}$$

$\alpha + i\beta = -\frac{5}{6} + \frac{i}{4}$ is not a root $\Rightarrow$ $\underline{s=0}$

$$Y_p(t) = e^{-5t/6}\left(A_1 \cos\frac{t}{4} + A_2 \sin\frac{t}{4}\right)$$

(8) $3y'' + 5y' - 2y = 3e^{2t}\cos 10t$

We find a particular solution in the form

$$Y_p(t) = y(t) = A_1 e^{2t}\cos 10t + A_2 e^{2t}\sin 10t \Rightarrow$$

$$y'(t) = 2A_1 e^{2t}\cos 10t - 10A_1 e^{2t}\sin 10t + 2A_2 e^{2t}\sin 10t +$$
$$+ 10A_2 e^{2t}\cos 10t = (2A_1 + 10A_2)e^{2t}\cos 10t + (-10A_1 + 2A_2) \times$$
$$e^{2t}\sin 10t$$

$$y''(t) = (4A_1 + 20A_2)e^{2t}\cos 10t - (20A_1 + 100A_2)e^{2t}\sin 10t +$$
$$+ (-20A_1 + 4A_2)e^{2t}\sin 10t + (-100A_1 + 20A_2)e^{2t}\cos 10t =$$
$$= (-96A_1 + 40A_2)e^{2t}\cos 10t - (40A_1 + 96A_2)e^{2t}\sin 10t$$

$$3y'' + 5y' - 2y = (-288A_1 + 120A_2 + 10A_1 + 50A_2 - 2A_1)e^{2t}\cos 10t$$
$$+ (-120A_1 - 288A_2 - 50A_1 + 10A_2 - 2A_2)e^{2t}\sin 10t =$$
$$= (-280A_1 + 170A_2)e^{2t}\cos 10t + (-170A_1 - 280A_2)e^{2t}\sin 10t =$$

$= 3e^{2t}\cos 10t \Rightarrow$

$$\begin{cases} -280A_1 + 170A_2 = 3 & \times 28 \quad \text{Eq 1} \\ -170A_1 - 280A_2 = 0 & \times 17 \quad \text{Eq 2} \end{cases}$$

Eliminate A_2: $28\,\text{Eq 1} + 17\,\text{Eq 2} = (-7840 - 2890)A_1 = 84 \Rightarrow$

$-10730\,A_1 = 84 \Rightarrow$

$A_1 = -\frac{84}{10730} = -\frac{42}{5365}$

$A_2 = -\frac{17}{28} A_1 = \frac{17}{28} \cdot \frac{42}{5365} = \frac{51}{10730}$

A particular solution is

$Y_p(t) = -\frac{42}{5365} e^{2t}\cos 10t + \frac{51}{10730} e^{2t}\sin 10t$

From item (iii) $r_1 = \frac{1}{3}$, $r_2 = -2 \Rightarrow$ gen. solution is

$$\boxed{y(t) = -\frac{42}{5365} e^{2t}\cos 10t + \frac{51}{10730} e^{2t}\sin 10t + c_1 e^{\frac{1}{3}t} + c_2 e^{-2t}}$$

(c) $y'' + \omega_0^2 y = \cos\omega t + 2\sin\omega t$

Case 1 $\omega^2 \neq \omega_0^2$

Then we look for a particular solution in the form

$Y_p(t) = y(t) = A_1\cos\omega t + A_2\sin\omega t$

$y'(t) = \omega A_2\cos\omega t - \omega A_1\sin\omega t$

$y''(t) = -\omega^2 A_1\cos\omega t - \omega^2 A_2\sin\omega t$

$y''(t) + \omega_0^2 y = (\omega_0^2 - \omega^2)A_1\cos\omega t + (\omega_0^2 - \omega^2)A_2\sin\omega t = \cos\omega t +$

$+ 2\sin\omega t \Rightarrow$ $(\omega_0^2 - \omega^2)A_1 = 1 \Rightarrow A_1 = \frac{1}{\omega_0^2 - \omega^2}$

$(\omega_0^2 - \omega^2)A_2 = 2 \Rightarrow A_2 = \frac{2}{\omega_0^2 - \omega^2}$ $\Rightarrow$

A particular solution is

$$Y_p(t) = \frac{\cos\omega t}{\omega_0^2-\omega^2} + \frac{2\sin\omega t}{\omega_0^2-\omega^2} \Rightarrow$$

The general solution is

$$\boxed{y(t) = \frac{\cos\omega t}{\omega_0^2-\omega^2} + \frac{2\sin\omega t}{\omega_0^2-\omega^2} + C_1\cos\omega_0 t + C_2\sin\omega_0 t}$$

<u>Case 2</u> $\omega^2 = \omega_0^2$

Then we look for a particular solution in the form

$$Y_p(t) = y(t) = A_1 t\cos\omega_0 t + A_2 t\sin\omega_0 t$$

$$y'(t) = A_1\cos\omega_0 t + A_2\sin\omega_0 t - \omega_0 A_1 t\sin\omega_0 t + \omega_0 A_2 t\cos\omega_0 t$$

$$y''(t) = -\omega_0 A_1\sin\omega_0 t + \omega_0 A_2\cos\omega_0 t - \omega_0 A_1\sin\omega_0 t -$$

$$+ \omega_0 A_2\cos\omega_0 t - \omega_0^2 A_1 t\cos\omega_0 t - \omega_0^2 A_2 t\sin\omega_0 t =$$

$$= 2\omega_0 A_2\cos\omega_0 t - 2\omega_0 A_1\sin\omega_0 t - \omega_0^2 A_1 t\cos\omega_0 t -$$

$$-\omega_0^2 A_2 t\sin\omega_0 t$$

$$y'' + \omega_0^2 y = 2\omega_0 A_2\cos\omega_0 t - 2\omega_0 A_1\sin\omega_0 t = \cos\omega_0 t + 2\sin\omega_0 t$$

$$\Rightarrow \begin{cases} 2\omega_0 A_2 = 1 \\ -2\omega_0 A_1 = 2 \end{cases} \quad \begin{aligned} &\Rightarrow A_2 = \frac{1}{2\omega_0} \\ &A_2 = -\frac{1}{\omega_0} \Rightarrow \end{aligned}$$

$$Y_p(t) = t\left(-\frac{1}{\omega_0}\cos\omega_0 t + \frac{1}{2\omega_0}\sin\omega_0 t\right) \Rightarrow$$

The gen. solution is

$$y(t) = t\left(-\frac{1}{\omega_0}\cos\omega_0 t + \frac{1}{2\omega_0}\sin\omega_0 t\right) + C_1\cos\omega_0 t + C_2\sin\omega_0 t$$