Homework Assignment 7 in Differential Equations, MATH308-SPRING 2017

due April 11, 2017 *Topics covered: The case of complex eigenvalues(section 7.6),Matrix exponential (section 7.7), Repeated eigenvalues (the case of $n = 2$)(section 7.8)*

1. Given the following system of linear differential equations:

$$\begin{cases} x_1' &= 4x_1 - 2x_2 \\ x_2' &= 4x_1 + 8x_2 \end{cases} \tag{1}$$

   (a) Find the general solution of the system (3).

   (b) If $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ is a solution of (3), what is the limit of $x(t)$ as $t \to +\infty$. Does this limit depend on initial conditions?

   (c) Find the solution of the system (3) satisfying the initial conditions: $x_1(0) = -3, \quad x_2(0) = 5$.

2. Given the following system of linear differential equations:

$$\begin{cases} x_1' &= -11x_1 - 6x_2 + 2x_3 \\ x_2' &= 14x_1 + 9x_2 - 2x_3 \\ x_3' &= -12x_1 - 7x_2 + x_3 \end{cases} \tag{2}$$

   (a) It is known that 3 is an eigenvalue of the corresponding matrix. Find the general solution of the system (2).

   (b) Find all $\alpha_1, \alpha_2, \alpha_3$ such that if $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$ is the solution of the system (2) with initial condition $x(0) = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$ then $x(t) \to 0$ as $t \to +\infty$.

   (c) Find all $\beta_1, \beta_2, \beta_3$ such that if $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$ is the solution of the system (2) with initial condition $x(0) = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}$ then $x(t) \to 0$ as $t \to -\infty$.

3. (a) $N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Calculate e^{tN} using the definition of the matrix exponential. (Hint: Certain power of N vanish

   (b) Let $A = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$ Calculate e^{tA} (Hint: use that $A = \lambda I + N$, the previous item, and the fact that if $AB = BA$, then $e^{A+B} = e^A e^B$)

4. Given the following system of linear differential equations:

$$\begin{cases} x_1' &= -3x_1 + x_2 \\ x_2' &= -x_1 - 5x_2 \end{cases} \tag{3}$$

   (a) Find the general solution of the system (3).

   (b) If $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ is a solution of (3), what is the limit of $x(t)$ as $t \to +\infty$. Does this limit depend on initial conditions?

   (c) Find the solution of the system (3) satisfying the initial conditions: $x_1(0) = 2, \quad x_2(0) = -1$.