Page 1) Solution of Homework #7 MATH 308

Problem 1

a) $\begin{cases} x_1' = 4x_1 - 2x_2 \\ x_2' = 4x_1 + 8x_2 \end{cases} \Rightarrow A = \begin{pmatrix} 4 & -2 \\ 4 & 8 \end{pmatrix}$

1) Eigenvalues

Char. equation $\lambda^2 - \operatorname{tr}A\,\lambda + \det A = \lambda^2 - 12\lambda + 40 = 0$

$\operatorname{tr}A = 12$

$\det A = 32 - (-8) = 40$

$D = 144 - 160 = -16$

$\lambda_{1,2} = \frac{12 \pm 4i}{2} = 6 \pm 2i$

2) An eigenvector of $\lambda = 6 + 2i$

$$(A - (6+2i)I)v = \begin{pmatrix} 4-(6+2i) & -2 \\ 4 & 8-(6+2i) \end{pmatrix} v = \begin{pmatrix} -2-2i & -2 \\ 4 & 2-2i \end{pmatrix} v = 0$$

$\Leftrightarrow (-2-2i)v_1 - 2v_2 = 0 \Leftrightarrow (1+i)v_1 + v_2 = 0$

Take $v_1 = 1 \Rightarrow v_2 = -1-i \Rightarrow v = \begin{pmatrix} 1 \\ -1-i \end{pmatrix}$

A complex-valued solution is

$$e^{(6+2i)t}\begin{pmatrix} 1 \\ -1-i \end{pmatrix} = e^{6t}(\cos 2t + i\sin 2t)\begin{pmatrix} 1 \\ -1-i \end{pmatrix} =$$

$$= e^{6t}\begin{pmatrix} \cos 2t + i\sin 2t \\ -\cos 2t + \sin 2t + i(-\sin 2t - \cos 2t) \end{pmatrix} = \underbrace{e^{6t}\begin{pmatrix} \cos 2t \\ -\cos 2t + \sin 2t \end{pmatrix}}_{\text{Real part}} +$$

$$i\underbrace{\begin{pmatrix} \sin 2t \\ -\sin 2t - \cos 2t \end{pmatrix}}_{\text{Imaginary part}} \Rightarrow \text{gen. solution is}$$

$$x(t) = e^{6t}\left(C_1\begin{pmatrix} \cos 2t \\ -\cos 2t + \sin 2t \end{pmatrix} + C_2\begin{pmatrix} \sin 2t \\ -\sin 2t - \cos 2t \end{pmatrix}\right)$$

$$= e^{6t}\begin{pmatrix} C_1\cos 2t + C_2\sin 2t \\ -(C_1+C_2)\cos 2t + (C_1 - C_2)\sin 2t \end{pmatrix}$$

(B) The length $|X(t)| \xrightarrow[t\to+\infty]{} \infty$ unless $C_1 = C_2 = 0$

i.e unless $X(0)$ = the origin. In the latter case $X(t) \equiv 0$

so $X(t) \xrightarrow[t\to+\infty]{} 0$. In this sense the answer depends on

the initial conditions

(c) $\begin{matrix} x_1(0) = -3 \\ x_2(0) = 5 \end{matrix} \Rightarrow \quad C_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ 5 \end{pmatrix} \Rightarrow$

$C_1 = -3$

$-C_1 - C_2 = 5 \Rightarrow C_2 = -C_1 - 5 = 3 - 5 = -2$

$\Rightarrow X(t) = e^{6t} \left(-3 \begin{pmatrix} \cos 2t \\ -\cos 2t + \sin 2t \end{pmatrix} - 2 \begin{pmatrix} \sin 2t \\ -\sin 2t - \cos 2t \end{pmatrix} \right)$

$= e^{6t} \begin{pmatrix} -3\cos 2t - 2\sin 2t \\ 5\cos 2t - \sin 2t \end{pmatrix}$

<u>Problem 2</u> $\begin{cases} x_1' = -11x_1 - 6x_2 + 2x_3 \\ x_2' = 14x_1 + 9x_2 - 2x_3 \\ x_3' = -12x_1 - 7x_2 + x_3 \end{cases}$

$A = \begin{pmatrix} -11 & -6 & 2 \\ 14 & 9 & -2 \\ -12 & -7 & 1 \end{pmatrix}$

1) Eigenvalues : $\det(A - \lambda I) = \begin{vmatrix} -11-\lambda & -6 & 2 \\ 14 & 9-\lambda & -2 \\ -12 & -7 & 1-\lambda \end{vmatrix} =$

$= -(\lambda+11)\Big(\underbrace{(\lambda-9)(\lambda-1) - 14}_{\lambda^2 - 10\lambda - 5} \Big) + 6 \Big(\underbrace{14(1-\lambda) - 24}_{-10-14\lambda} \Big) + 2 \Big(\underbrace{-98 + 12(9-\lambda)}_{10 - 12\lambda} \Big) =$

$= -\lambda^3 + 10\lambda^2 + 5\lambda - 11\lambda^2 + 110\lambda + 55 - 60 - 84\lambda + 20 - 24\lambda =$

$= -\lambda^3 - \lambda^2 + 7\lambda + 15$

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Check for integer roots: if they exist they must be factors of 15; i.e. $\pm1, \pm3, \pm5, \pm15$

Plug: $\lambda = 1$: $-1-1+7+15 \neq 0$ ✗

$\lambda=-1$: $1-1-7+15 \neq 0$ ✗

$\lambda = 3$: $-27-9+21+15 = -36+36 = 0$ ✓

$\lambda = 3$ is an eigenvalue and $(\lambda-3)$ divides $\lambda^3+\lambda^2-7\lambda-15$

$$\begin{array}{r|l} & \lambda^2+4\lambda+5 \\ \hline \lambda-3 & \lambda^3+\lambda^2-7\lambda-15 \\ & \lambda^3-3\lambda^2 \\ \hline & 4\lambda^2-7\lambda \\ & 4\lambda^2-12\lambda \\ \hline & 5\lambda-15 \end{array}$$

$\Rightarrow \lambda^2+4\lambda+5=0$

$D = 16-20=-4$

$\lambda_{1,2} = \frac{-4\pm 2i}{2} = -2\pm i$

So $\lambda_{1,2} = -2\pm i$, $\lambda_3 = 3$

2) Eigenvectors of $\lambda_{1,2} = -2+i$

$$(A-(-2+i)I)v = (A+(2-i)I)v = \begin{pmatrix} -9-i & -6 & 2 \\ 14 & 11-i & -2 \\ -12 & -7 & 3-i \end{pmatrix} v = 0$$

Augmented matrix: → use this as a pivot

$$\left(\begin{array}{ccc|c} -9-i & -6 & 2 & 0 \\ 14 & 11-i & -2 & 0 \\ -12 & -7 & 3-i & 0 \end{array}\right) \xrightarrow[R_3 \to 2R_3-(3-i)R_1]{R_2 \to R_2+R_1} \left(\begin{array}{ccc|c} -9-i & -6 & 2 & 0 \\ 5-i & 5-i & 0 & 0 \\ 4-6i & 4-6i & 0 & 0 \end{array}\right) \xrightarrow[R_3 \to \frac{1}{4-6i}R_3]{R_2 \to \frac{1}{5-i}R_2} \left(\begin{array}{ccc|c} -9-i & -6 & 2 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right)$$

$-14+6(3-i) = 4-6i$

$-24-(3-i)(-9-i) = -24-(-27+6i-1) = 4-6i$

$$\xrightarrow{R_3 \to R_3 - R_2} \left(\begin{array}{ccc|c} -9-i & -6 & 2 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{l} (-9-i)v_1 - 6v_2 + 2v_3 = 0 \\ v_1 + v_2 = 0 \end{array}$$

Take $v_2 = 1 \Rightarrow v_1 = -1 \Rightarrow$ (plug into the 1st equation)

$+9+i-6+2v_3=0 \Rightarrow 2v_3=-3-i \Rightarrow v_3=-\frac{3}{2}-\frac{i}{2}$

$$\Rightarrow V=\begin{pmatrix}-1\\1\\-\frac{1}{2}(3+i)\end{pmatrix}$$ $\rightarrow$ Rem: you can take any complex multiple of it as well

Or maybe in order not to have denominator multiply v by -2 to get $v=\begin{pmatrix}2\\-2\\3+i\end{pmatrix}$

Calculate: $e^{(-2+i)t}\begin{pmatrix}2\\-2\\3+i\end{pmatrix}=e^{-2t}(\cos t+i\sin t)\begin{pmatrix}2\\-2\\3+i\end{pmatrix}=$

$$=e^{-2t}\begin{pmatrix}2\cos t+i2\sin t\\-2\cos t-i2\sin t\\3\cos t-\sin t+i(3\sin t+\cos t)\end{pmatrix}=e^{-2t}\begin{pmatrix}2\cos t\\-2\cos t\\3\cos t-\sin t\end{pmatrix}+$$

$$+i\begin{pmatrix}2\sin t\\-2\sin t\\3\sin t+\cos t\end{pmatrix}\Rightarrow x^1(t)=e^{-2t}\begin{pmatrix}2\cos t\\-2\cos t\\3\cos t-\sin t\end{pmatrix}$$ and

$$x^2(t)=e^{-2t}\begin{pmatrix}2\sin t\\-2\sin t\\3\sin t+\cos t\end{pmatrix}$$ are 2 indep. solutions

3) Eigenvector of $\lambda=3$

$$(A-3I)v=\begin{pmatrix}-14&-6&2\\14&6&-2\\-12&-7&2\end{pmatrix}v=0$$

Augmented matrix

$$\left(\begin{array}{ccc|c} -14 & -6 & 2 & 0 \\ 14 & 6 & -2 & 0 \\ -12 & -7 & -2 & 0 \end{array}\right) \underset{R_1 \leftrightarrow R_3}{\sim} \left(\begin{array}{ccc|c} -12 & -7 & -2 & 0 \\ 14 & 6 & -2 & 0 \\ -14 & -6 & 2 & 0 \end{array}\right) \xrightarrow{R_3 \to R_3+R_2} \left(\begin{array}{ccc|c} -12 & -7 & -2 & 0 \\ 14 & 6 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_2 \to R_1 - R_2}$$

$$\left(\begin{array}{ccc|c} -12 & -7 & -2 & 0 \\ 26 & 13 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \underset{R_1 \to -R_1}{\xrightarrow{R_2 \to \frac{1}{13} R_2}} \left(\begin{array}{ccc|c} 12 & 7 & 2 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{l} 12v_1 + 7v_2 + 2v_3 = 0 \\ 2v_1 + v_2 = 0 \end{array}$$

triangle here

Take $v_1 = 1 \Rightarrow v_2 = -2 \Rightarrow$

$12 - 14 + 2v_3 = 0 \Rightarrow v_3 = 1$

$\Rightarrow \tilde{v} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ is an eigenvector of $\lambda = 3 \Rightarrow x^3(t) = e^{3t} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

is a third solution independent of 2 solution that were already found. $\Rightarrow$ the general solution is

$$x(t) = e^{-2t} \left(c_1 \begin{pmatrix} 2\cos t \\ -2\sin t \\ 3\cos t - \sin t \end{pmatrix} + c_2 \begin{pmatrix} 2\sin t \\ -2\sin t \\ 3\sin t + \cos t \end{pmatrix} \right) +$$

$$+ c_3 e^{3t} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

Problem 2

(b) $\quad x(t) \underset{t \to +\infty}{\longrightarrow} 0 \Leftrightarrow c_3 = 0 \Rightarrow x(0) = c_1 \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

i.e. $\begin{cases} \alpha_1 = 2c_1 \\ \alpha_2 = -2c_1 \\ \alpha_3 = 3c_1 + c_2 \end{cases} \to$ a plane $\Leftrightarrow \boxed{\begin{array}{l} \alpha_1 = -\alpha_2 \Leftrightarrow \\ \alpha_1 + \alpha_2 = 0 \end{array}}$

More explanations:
This is a plane parallel to vector $\begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and passing through the origin

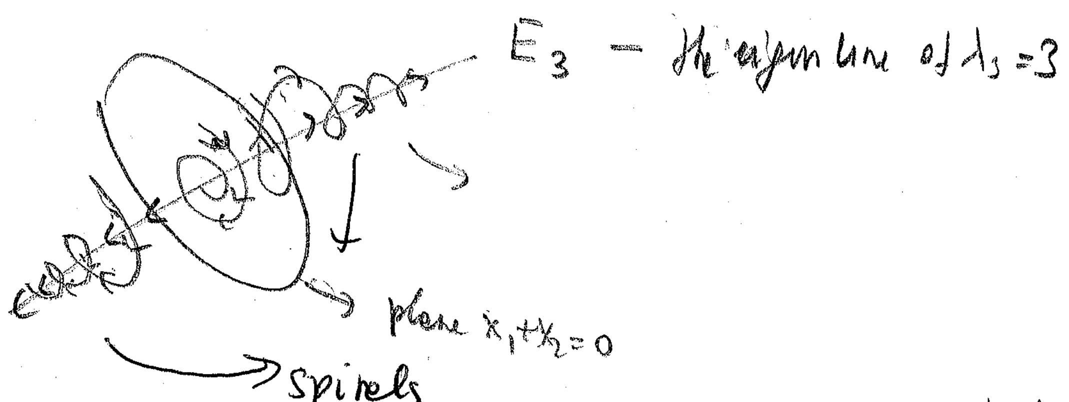
you can take cross-product of these to vectors to get a normal to this plane $\begin{pmatrix} i & j & k \\ 2 & -2 & 3 \\ 0 & 0 & 1 \end{pmatrix} = -2i-2j \Rightarrow$ the plane is orthogonal to vector $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \Rightarrow$

It has equation $\alpha_1+\alpha_2=0$

(c) $x(t) \xrightarrow[t\to-\infty]{} 0 \iff C_1=C_2=0$ i.e. $x(0)=C_3\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \text{Const}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

Visualization
<u>The phase portrate</u> here (in 3-dim. space)



<u>Problem 3</u> (a) By definition of the matrix exponential
$$e^{tN} = I+tN+\frac{t^2}{2}N^2+\frac{t^3}{3!}N^3+\frac{t^4}{4!}N^4+\dots$$

Let us study the powers N^i of matrix N

$$N^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$N^3 = N^2 N = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

So $N^3 = 0 \Rightarrow N^k = 0$ for any $k \geq 3 \Rightarrow$

$$\boxed{e^{tN} = I + tN + \frac{t^2}{2} N^2} = \boxed{\begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}}$$

(b) $e^{tA} = \underbrace{e^{t(\lambda I + N)}}_{\lambda I \text{ commutes with } N} = \underbrace{e^{\lambda t I}}_{e^{\lambda t} I} e^{tN} = e^{\lambda t} e^{tN} =$

$$= e^{\lambda t}\left(I + tN + \frac{t^2}{2} N^2\right) = \boxed{e^{\lambda t} \begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}}$$

<u>Problem 4</u> $\begin{cases} x_1' = -3x_1 + x_2 \\ x_2' = -x_1 - 5x_2 \end{cases}$

$A = \begin{pmatrix} -3 & 1 \\ -1 & -5 \end{pmatrix}$

1) Eigenvalues $\det(A - \lambda I) = \lambda^2 - \operatorname{tr} A \lambda + \det A =$

$\operatorname{tr} A = -8$

$\det A = 15 + 1 = 16$

$= \lambda^2 + 8\lambda + 16 = (\lambda + 4)^2 = 0 \Rightarrow$

$\lambda_{1,2} = -4$ is a repeated root $\Rightarrow$

2) algebraic multiplicity of $\lambda = -4$ is equal to 2

Since $A \neq -4I$ we know that in this case the geometric multiplicity is equal to 1.

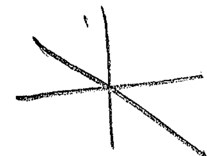
I will demonstrate the solution using each of 3 algorithms given in the class

<u>Using algorithm 1</u> a) Find the eigenline E_{-4} by solving

$$(A+4I)v=0$$

$$(A+4I)v = \begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}=0 \iff v_1+v_2=0 \Rightarrow v_1=-v_2$$

$$E_4 = \left\{c\begin{pmatrix}1\\ -1\end{pmatrix},\ c\in\mathbb{R}\right\}$$



(b) take any $w \in \mathbb{R}^2$ which is not in E_4, for example,

$w=\begin{pmatrix}1\\ 0\end{pmatrix}$ and let $v=(A+4I)\begin{pmatrix}1\\ 0\end{pmatrix} = \begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}=\begin{pmatrix}1\\ -1\end{pmatrix}$

$$\Rightarrow \left\{e^{-4t}v,\ e^{-4t}(w+tv)\right\} = \left\{e^{-4t}\begin{pmatrix}1\\ -1\end{pmatrix},\ e^{-4t}\left(\begin{pmatrix}1\\ 0\end{pmatrix}+t\begin{pmatrix}1\\ -1\end{pmatrix}\right)\right\}$$

is a fundamental set of solutions $\Rightarrow$ the general solution

$$x(t) = e^{-4t}\left(c_1\begin{pmatrix}1\\ -1\end{pmatrix} + c_2\left(\begin{pmatrix}1\\ 0\end{pmatrix}+t\begin{pmatrix}1\\ -1\end{pmatrix}\right)\right) =$$

$$= \boxed{e^{-4t}\begin{pmatrix}c_1+c_2+c_2t\\ -c_1-c_2t\end{pmatrix}}$$

<u>Using algorithm 2</u> Choose some eigenvector for example $v=\begin{pmatrix}1\\ -1\end{pmatrix}$ and find w s.t.

$$(A+4I)w = v$$

$$\begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix} w = \begin{pmatrix}1\\ -1\end{pmatrix} \iff w_1 + w_2 = 1$$

Take $w_2 = 0 \Rightarrow w_1 = 1 \Rightarrow w = \begin{pmatrix}1\\ 0\end{pmatrix}$

Then the rest is exactly the same as in the previous algorithm.

Using algorithm 3 For this we use that in our situation $(A+4I)^2 = 0 \Rightarrow$

$$e^{tA} = e^{-4t} e^{t(A+4I)} = e^{-4t}\left(I + t(A+4I)\right) =$$

$$= e^{-4t}\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + t\begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix}\right) = e^{-4t}\begin{pmatrix}1+t & t\\ -t & 1-t\end{pmatrix}$$

$\Rightarrow$ columns of this matrix form a fundamental set of solutions i.e. the general solution is

$$x(t) = e^{-4t}\left(\tilde{c}_1\begin{pmatrix}1+t\\ -t\end{pmatrix} + \tilde{c}_2\begin{pmatrix}t\\ 1-t\end{pmatrix}\right) = e^{-4t}\begin{pmatrix}\tilde{c}_1 + (\tilde{c}_1+\tilde{c}_2)t\\ \tilde{c}_2 - (\tilde{c}_1+\tilde{c}_2)t\end{pmatrix}$$

Remark: The relation between (c_1, c_2) from algorithm 1 and $(\tilde{c}_1, \tilde{c}_2)$ from algorithm 3 is

$$c_2 = \tilde{c}_1 + \tilde{c}_2$$
$$c_1 = -\tilde{c}_2$$

Problem 4 (b)

Useing the form of the answer as in Algorithm 1

$C_1 + C_2 = 2$

$-C_1 = -1 \Rightarrow C_1 = 1 \Rightarrow C_2 = 1 \Rightarrow$

$$\boxed{x(t) = e^{-4t}\begin{pmatrix} 2+t \\ -1-t \end{pmatrix}}$$