Homework #8 MATH 308 Solutions

Problem 1

$$\begin{cases} x_1' = -2x_1 + 3x_2 - x_3 \\ x_2' = 4x_1 - x_2 + x_3 \\ x_3' = -4x_1 - 3x_2 - 5x_3 \end{cases}$$

(a) $A = \begin{pmatrix} -2 & 3 & -1 \\ 4 & -1 & 1 \\ -4 & -3 & -5 \end{pmatrix}$

We know that $\lambda = 2, -4, -6$ are the eigenvalues
The eigenvalues are distinct.
Find the corresponding eigenvectors

1) For $\lambda = 2$

$$(A - 2I)v = \begin{pmatrix} -4 & 3 & -1 \\ 4 & -3 & 1 \\ -4 & -3 & -7 \end{pmatrix} v = 0$$

$$\left(\begin{array}{ccc|c} -4 & 3 & -1 & 0 \\ 4 & -3 & 1 & 0 \\ -4 & -3 & -7 & 0 \end{array}\right) \xrightarrow{R_3 \leftrightarrow R_1} \left(\begin{array}{ccc|c} -4 & -3 & -7 & 0 \\ 4 & -3 & 1 & 0 \\ -4 & 3 & -1 & 0 \end{array}\right) \xrightarrow[R_3 \to R_3 + R_2]{R_2 \to R_2 + R_1}$$

$$\left(\begin{array}{ccc|c} -4 & -3 & -7 & 0 \\ 0 & -6 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow$$

$-4v_1 - 3v_2 - 7v_3 = 0$ (Eq 1)

$-6v_2 - 6v_3 = 0 \Rightarrow v_2 + v_3 = 0$ (Eq 2)

Take $v_3 = 1 \Rightarrow v_2 = -1$

$-4v_1 + 3 - 7 = 0 \Rightarrow -4v_1 = 4 \Rightarrow v_1 = -1$

$\Rightarrow v^1 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$ is an eigenvector of $\lambda = 2 \Rightarrow e^{2t}\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$ is a solution of our system

2) For $\lambda = -4$

$$(A-(-4)I)v = (A+4I)v = \begin{pmatrix} 2 & 3 & -1 \\ 4 & 3 & 1 \\ -4 & -3 & -1 \end{pmatrix} v = 0$$

Augmented matrix:

$$\left(\begin{array}{ccc|c} 2 & 3 & -1 & 0 \\ 4 & 3 & 1 & 0 \\ -4 & -3 & -1 & 0 \end{array}\right) \underset{R_3 \to R_3 + R_1}{\overset{R_2 \to R_2 - 2R_1}{\sim}} \left(\begin{array}{ccc|c} 2 & 3 & -1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$2v_1 + 3v_2 - v_3 = 0$ (Eq 1)

$-3v_2 + 3v_3 = 0 \Rightarrow -v_2 + v_3 = 0$

Take $v_3 = 1 \Rightarrow v_2 = 1 \overset{(Eq 1)}{\Rightarrow} 2v_1 + 3 - 1 = 0 \Rightarrow 2v_1 + 2 = 0 \Rightarrow v_1 = -1 \Rightarrow$

$v^2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector of $\lambda = -4 \Rightarrow e^{-4t}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ is a solution of our system

3) For $\lambda = -6$

$$(A-(-6)I)v = (A+6I)v = \begin{pmatrix} 4 & 3 & -1 \\ 4 & 5 & 1 \\ -4 & -3 & 1 \end{pmatrix} v = 0$$

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Augmented matrix

$$\left(\begin{array}{ccc|c} 4 & 3 & -1 & 0 \\ 4 & 5 & 1 & 0 \\ -4 & -3 & 1 & 0 \end{array}\right) \underset{R_3 \to R_3 + R_1}{\overset{R_2 \to R_2 - R_1}{\sim}} \left(\begin{array}{ccc|c} 4 & 3 & -1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$4v_1 + 3v_2 - v_3 = 0$ (Eq 1)

$2v_2 + 2v_3 = 0 \Rightarrow v_2 = -v_3$. Take $v_3 = 1 \Rightarrow v_2 = -1$

$\overset{Eq\,1}{\Rightarrow}$ $4v_1 - 3 - 1 = 0 \Rightarrow v_1 = 1 \Rightarrow$

$v^3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ is an eigenvector of $\lambda = -6 \Rightarrow e^{-6t}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ is a solution.

Combining all together

$$x(t) = c_1 e^{2t}\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} + c_2 e^{-4t}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + c_3 e^{-6t}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

is the general solution.

(B) We have to solve

$$c_1\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + c_3\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -7 \\ 9 \\ -10 \end{pmatrix}$$

Augmented matrix

$$\left(\begin{array}{ccc|c} -1 & -1 & 1 & -7 \\ -1 & 1 & -1 & 9 \\ 1 & 1 & 1 & -10 \end{array}\right) \underset{R_3 \to R_3 + R_1}{\overset{R_2 \to R_2 - R_1}{\sim}} \left(\begin{array}{ccc|c} -1 & -1 & 1 & -7 \\ 0 & 2 & -2 & 16 \\ 0 & 0 & 2 & -17 \end{array}\right) \Rightarrow$$

$$\begin{aligned} -c_1 - c_2 + c_3 &= -7 \\ c_2 - c_3 &= 8 \\ 2c_3 &= -17 \end{aligned}$$

Backward substitution $\boxed{c_3 = -\frac{17}{2}} \Rightarrow$

$\Rightarrow c_2 + \frac{17}{2} = 8 \Rightarrow \boxed{c_2} = 8 - \frac{17}{2} = \boxed{-\frac{1}{2}} \Rightarrow$

$-c_1 + \frac{1}{2} - \frac{17}{2} = -7 \Rightarrow -c_1 = -7 + 8 \Rightarrow$

$\boxed{c_1 = -1}$

$$\Rightarrow \boxed{X(t) = -e^{2t}\begin{pmatrix}-1\\-1\\1\end{pmatrix} - \frac{1}{2}e^{-4t}\begin{pmatrix}-1\\1\\1\end{pmatrix} - \frac{17}{2}e^{-6t}\begin{pmatrix}1\\-1\\1\end{pmatrix}}$$

Problem 2 $x_1' = -7x_1 + 5x_2$

$x_2' = -4x_1 + x_2$

1a) $A = \begin{pmatrix}-7 & 5\\-4 & 1\end{pmatrix}$

Characteristic equation $\lambda^2 - \underbrace{\operatorname{tr} A}_{(-6)}\lambda + \underbrace{\det A}_{13} = 0 \Rightarrow$

$\lambda^2 + 6\lambda + 13 = 0$

$D = 36 - 4\cdot 13 = 36 - 52 = -16$

$\lambda_{1,2} = \frac{-6 \pm 4i}{2} = -3 \pm 2i$

Eigenvector of $\lambda = -3+2i$

$(A - \lambda I)v = \begin{pmatrix}-7+3-2i & 5\\-4 & 1+3-2i\end{pmatrix}v = \begin{pmatrix}-4-2i & 5\\-4 & 4-2i\end{pmatrix}v = 0$

Note that the second equation is multiple of the first

(check (not necessary) $\frac{5}{4-2i} = \frac{5}{4+4}(4+2i) \Rightarrow \frac{4+2i}{4} = \frac{-4-2i}{-4}$

$\Rightarrow$ only one relation $(-4-2i)v_1 + 5v_2 = 0$

Take $v_1 = 1 \Rightarrow -4-2i + 5v_2 \Rightarrow v_2 = \frac{4+2i}{5}$

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$\Rightarrow v = \begin{pmatrix} 1 \\ \frac{4+2i}{5} \end{pmatrix}$ is an eigenvector of $\lambda = -3+2i$

Also $v = \begin{pmatrix} 5 \\ 4+2i \end{pmatrix}$ is an eigenvector of $\lambda = -3+2i$

$\Rightarrow$ a complex value solution is $e^{(-3+2i)t}\begin{pmatrix} 5 \\ 4+2i \end{pmatrix}$

Find its real and imaginary parts (then they will form the fundamental set of solutions

$$\to e^{-3t}(\cos 2t + i\sin 2t)\begin{pmatrix} 5 \\ 4+2i \end{pmatrix} =$$

$$= e^{-3t}\left(\begin{pmatrix} 5\cos 2t \\ 4\cos 2t - 2\sin 2t \end{pmatrix} + i\begin{pmatrix} 5\sin 2t \\ 4\sin 2t + 2\cos 2t \end{pmatrix}\right)$$

$\Rightarrow$ Gen solution is

$$x(t) = e^{-3t}\left(c_1\begin{pmatrix} 5\cos 2t \\ 4\cos 2t - 2\sin 2t \end{pmatrix} + c_2\begin{pmatrix} 5\sin 2t \\ 4\sin 2t + 2\cos 2t \end{pmatrix}\right)$$

(b) Since cos2t and sint are bounded, their linear combinations with constant coefficients are also bounded. Therefore since $\lim_{t\to+\infty} e^{-3t} = 0$

we get that $x(t) \to 0$

c) Substituting into the answer of 1a) $t=0$

$$c_1\begin{pmatrix} 5 \\ 4 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ -4 \end{pmatrix} \Rightarrow \begin{matrix} 5c_1 = 5 \Rightarrow c_1 = 1 \\ 4c_1 + 2c_2 = -4 \Rightarrow \end{matrix}$$

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$4+2C_2=-4 \Rightarrow 2C_2=-8 \Rightarrow C_2=-4$

$$\Rightarrow \quad X(t) = e^{-3t}\begin{pmatrix} 5\cos 2t - 20\sin 2t \\ 4\cos 2t - 2\sin 2t - 16\sin 2t - 8\cos 2t \end{pmatrix} =$$

$$= \boxed{e^{-3t}\begin{pmatrix} 5\cos 2t - 20\sin 2t \\ -4\cos 2t - 18\sin 2t \end{pmatrix}}$$

Problem 3

$$x_1' = -x_1 + 3x_2 - x_3$$
$$x_2' = -4x_1 + x_2 + 4x_3$$
$$x_3' = 2x_2 - x_3$$

(a) $A = \begin{pmatrix} -1 & 3 & -1 \\ -4 & 1 & 4 \\ 0 & 2 & -1 \end{pmatrix}$

$$\det(A-\lambda I) = \begin{vmatrix} -1-\lambda & 3 & -1 \\ -4 & 1-\lambda & 4 \\ 0 & 2 & -1-\lambda \end{vmatrix} = -(\lambda+1)\begin{vmatrix} 1-\lambda & 4 \\ 2 & -1-\lambda \end{vmatrix} -$$

$$-3\begin{vmatrix} -4 & 4 \\ 0 & -1-\lambda \end{vmatrix} - \begin{vmatrix} -4 & 1-\lambda \\ 0 & 2 \end{vmatrix} = -(\lambda+1)(\lambda^2-1-8) -$$

$$-12(\lambda+1)+8 = -(\lambda+1)(\lambda^2-9) - 12\lambda - 12 + 8 =$$

$$= -\lambda^3 - \lambda^2 + 9\lambda + 9 - 12\lambda - 4 = -\lambda^3 - \lambda^2 - 3\lambda + 5 = 0$$

$$\Leftrightarrow \quad \lambda^3 + \lambda^2 + 3\lambda - 5 = 0$$

It is given that $\lambda = 1$ is a root $\Rightarrow \lambda^3 + \lambda^2 + 3\lambda - 5$ is divisible by $\lambda - 1$

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So

$$\begin{array}{r|l} & \lambda^2+2\lambda+5 \\ \lambda-1 & \lambda^3+\lambda^2+3\lambda-5 \\ & -\lambda^3-\lambda^2 \\ \hline & 2\lambda^2+3\lambda \\ & -2\lambda^2-2\lambda \\ \hline & 5\lambda-5 \\ & -5\lambda-5 \\ \hline & 0 \end{array}$$

c) $\lambda^2+2\lambda+5=0$

$D = 4-20=-16$

$\lambda = \frac{-2\pm 4i}{2} = -1 \pm 2i$

d) Eigenvector of $\lambda = -1+2i$

$$(A-(1+4i)I)v = \begin{pmatrix} -1-(-1+2i) & 3 & -1 \\ -4 & 1-(-1+2i) & 4 \\ 0 & 2 & -1-(-1+2i) \end{pmatrix} v =$$

$$= \begin{pmatrix} -2i & 3 & -1 \\ -4 & 2-2i & 4 \\ 0 & 2 & -2i \end{pmatrix} v = 0$$

Augmented matrix

$$\left(\begin{array}{ccc|c} -2i & 3 & -1 & 0 \\ -4 & 2-2i & 4 & 0 \\ 0 & 2 & -2i & 0 \end{array}\right) \underset{R_2\to iR_2-2R_1}{\sim} \left(\begin{array}{ccc|c} -2i & 3 & -1 & 0 \\ 0 & 2i+2-6 & 4i+2 & 0 \\ 0 & 2 & -2i & 0 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|c} -2i & 3 & -1 & 0 \\ 0 & -4+2i & 2+4i & 0 \\ 0 & 2 & -2i & 0 \end{array}\right) \underset{\substack{R_2\to\frac{1}{2}R_2 \\ R_3\to\frac{1}{2}R_3 \\ R_2\leftrightarrow R_3}}{\sim} \left(\begin{array}{ccc|c} -2i & 3 & -1 & 0 \\ 0 & 1 & -i & 0 \\ 0 & -2+i & 1+2i & 0 \end{array}\right) \sim$$

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$R_3 \to R_3 - (-2+i)R_2$ $\sim$
$$\left(\begin{array}{ccc|c} -2i & 3 & -1 & 0 \\ 0 & 1 & -i & 0 \\ 0 & 0 & \underline{0} & 0 \end{array}\right) \Rightarrow$$

$1+2i+i(-2+i)=0$

$-2iv_1+3v_2-v_3=0$ (Eq 1)

$v_2 - iv_3 = 0 \Rightarrow$ Take $v_3=1 \Rightarrow v_2 = i \Rightarrow$ (Eq 1)

$-2iv_1+3i-1=0 \Rightarrow -2iv_1 = 1-3i \Rightarrow$

$v_1 = \frac{3i-1}{2i} = \frac{3}{2} - \frac{1}{2i} = \frac{3}{2}+\frac{1}{2}i \Rightarrow$

$\begin{pmatrix} \frac{3}{2}+\frac{i}{2} \\ i \\ 1 \end{pmatrix}$ or $\begin{pmatrix} 3+i \\ 2i \\ 2 \end{pmatrix}$ are eigenvectors of $-1+2i$

$\Rightarrow e^{(-1+2i)t}\begin{pmatrix} 3+i \\ 2i \\ 2 \end{pmatrix}$ is a complex-valued solution

Find the real part and the imaginary part to get 2 independent real solutions:

$$e^{-t}(\cos 2t + i\sin 2t)\begin{pmatrix} 3+i \\ 2i \\ 2 \end{pmatrix} = e^{-t}\begin{pmatrix} 3\cos 2t - \sin 2t \\ -2\sin 2t \\ 2\cos 2t \end{pmatrix} +$$

$$+ie^{-t}\begin{pmatrix} \cos 2t + 3\sin 2t \\ 2\cos 2t \\ 2\sin 2t \end{pmatrix} \Rightarrow e^{-t}\begin{pmatrix} 3\cos 2t - \sin 2t \\ -2\sin 2t \\ 2\cos 2t \end{pmatrix} \text{ and}$$

$$e^{-t}\begin{pmatrix} \cos 2t + 3\sin 2t \\ 2\cos 2t \\ 2\sin 2t \end{pmatrix} \text{ are independent solutions}$$

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2) Eigenvector of $\lambda = 1$

$$(A - I)v = \begin{pmatrix} -2 & 3 & -1 \\ -4 & 0 & 4 \\ 0 & 2 & -2 \end{pmatrix} v = 0$$

Augmented matrix

$$\left(\begin{array}{ccc|c} -2 & 3 & -1 & 0 \\ -4 & 0 & 4 & 0 \\ 0 & 2 & -2 & 0 \end{array}\right) \underset{\sim}{R_2 \to R_2 - 2R_1} \left(\begin{array}{ccc|c} -2 & 3 & -1 & 0 \\ 0 & -6 & 6 & 0 \\ 0 & 2 & -2 & 0 \end{array}\right) \underset{\sim}{R_2 \to -\frac{1}{6} R_2}$$

$$\left(\begin{array}{ccc|c} -2 & 3 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & -2 & 0 \end{array}\right) \underset{\sim}{R_3 \to R_3 - 2R_2} \left(\begin{array}{ccc|c} -2 & 3 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow$$

$-2v_1 + 3v_2 - v_3 = 0$ (Eq 1)

$v_2 - v_3 = 0$. Take $v_3 = 1 \Rightarrow v_2 = 1 \overset{\text{Eq 1}}{\Longrightarrow} -2v_1 + 3 - 1 = 0 \Rightarrow$

$-2v_1 + 2 = 0 \Rightarrow v_1 = 1 \Rightarrow v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector of $\lambda = 1$

$\Rightarrow e^t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is a solution $\Rightarrow$

Combining 1) and 2), we get that the general solution is

$$x(t) = C_1 e^{-t} \begin{pmatrix} 3\cos 2t - \sin 2t \\ -2\sin 2t \\ 2\cos 2t \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} \cos 2t + 3\sin 2t \\ 2\cos 2t \\ 2\sin 2t \end{pmatrix} + C_3 e^t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(b) $$C_1 \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + C_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix}$$

Page 10) Augmented matrix

$$\left(\begin{array}{ccc|c} 3 & 1 & 1 & 4\\ 0 & 2 & 1 & -2\\ 2 & 0 & 1 & 3 \end{array}\right) \overset{R_3 \to 3R_3 - 2R_1}{\sim} \left(\begin{array}{ccc|c} 3 & 1 & 1 & 4\\ 0 & 2 & 1 & -2\\ 0 & -2 & 1 & 1 \end{array}\right) \overset{R_3 \to R_3 + R_2}{\sim}$$

$$\sim \left(\begin{array}{ccc|c} 3 & 1 & 1 & 4\\ 0 & 2 & 1 & -2\\ 0 & 0 & 2 & -1 \end{array}\right) \Rightarrow \begin{cases} 3c_1 + c_2 + c_3 = 4\\ 2c_2 + c_3 = -2\\ 2c_3 = -1 \end{cases}$$

Backward substitution: $\boxed{c_3 = -\frac{1}{2}} \Rightarrow 2c_2 - \frac{1}{2} = -2 \Rightarrow 2c_2 = -\frac{3}{2} \Rightarrow$

$\boxed{c_2 = -\frac{3}{4}} \Rightarrow 3c_1 - \frac{3}{4} - \frac{1}{2} = 4 \Rightarrow 3c_1 = 4 + \frac{5}{4} = \frac{21}{4} \Rightarrow$

$\boxed{c_1 = \frac{7}{4}} \Rightarrow$ substituting in the answer to a)

$$x(t) = e^{-t} \begin{pmatrix} \frac{7}{4}(3\cos 2t - \sin 2t) - \frac{3}{4}(\cos 2t + 3\sin 2t)\\ -\frac{7}{2}\sin 2t - \frac{3}{2}\cos 2t\\ \frac{7}{2}\cos 2t - \frac{3}{2}\sin 2t \end{pmatrix}$$

$$- \frac{1}{2} e^{t} \begin{pmatrix} 1\\ 1\\ 1 \end{pmatrix} = \boxed{e^{-t} \begin{pmatrix} \frac{9}{2}\cos 2t - 4\sin 2t\\ -\frac{7}{2}\sin 2t - \frac{3}{2}\cos 2t\\ \frac{7}{2}\cos 2t - \frac{3}{2}\sin 2t \end{pmatrix} - \frac{1}{2} e^{t} \begin{pmatrix} 1\\ 1\\ 1 \end{pmatrix}}$$

(c) $x(t) \to 0 \Leftrightarrow c_3 = 0 \Rightarrow$

$$x_0 = c_1 \begin{pmatrix} 3\\ 0\\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1\\ 2\\ 0 \end{pmatrix} = \begin{pmatrix} \alpha_1\\ \alpha_2\\ \alpha_3 \end{pmatrix} \Rightarrow \begin{cases} \alpha_1 = 3c_1 + c_2\\ \alpha_2 = 2c_2 \Rightarrow c_2 = \frac{1}{2}\alpha_2\\ \alpha_3 = 2c_1 \Rightarrow c_1 = \frac{1}{2}\alpha_3 \end{cases}$$

Substituting in the first relation: $\alpha_1 = \frac{3}{2}\alpha_3 + \frac{1}{2}\alpha_2 \Leftrightarrow$

$\boxed{2\alpha_1 - \alpha_2 - 3\alpha_3 = 0}$ — a plane in $\mathbb{R}^3$

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(d) $x(t) \xrightarrow[t \to +\infty]{} 0 \iff C_1 = C_2 = 0 \Rightarrow$

$X(0) = C_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} \Rightarrow \boxed{\beta_1 = \beta_2 = \beta_3}$