Homework assignment 8 Solution MATH 308-505

Problem 1

(a) $y'' + y = \cos^2 t$

The homogeneous eq. is $y''+y=0 \Rightarrow$
gen. solution of hom. eq. is $y(t) = C_1 \cos t + C_2 \sin t$

We a looking for solutions of nonhomogeneous equation in the form

$$y(t) = u_1(t)\cos t + u_2(t)\sin t$$

s.t $\begin{cases} \cos t\, u_1'(t) + \sin t\, u_2'(t) = 0 \\ -\sin t\, u_1'(t) + \cos t\, u_2'(t) = \cos^2 t \end{cases}$

$W(\cos t \sin t) = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = 1$

$\Downarrow$

$$u_1'(t) = \frac{\begin{vmatrix} 0 & \sin t \\ \cos^2 t & \cos t \end{vmatrix}}{W(\cos t, \sin t)} = -\cos^2 t \sin t$$

$$u_2'(t) = \frac{\begin{vmatrix} \cos t & 0 \\ -\sin t & \cos^2 t \end{vmatrix}}{W(\cos t, \sin t)} = \cos^3 t$$

$\Rightarrow u_1(t) = -\int \cos^2 t \sin t \, dt + C_1 = +\int u^2 du + C_1 = -\frac{u^3}{3} + C_1 =$

$u = \cos t$
$du = -\sin t\, dt$

$= \frac{\cos^3 t}{3} + C_1$

-2-

$$u_2(t) = \int \cos^3 t \, dt + C_2 = \int (1-u^2)\, du + C_2 = u - \frac{u^3}{3} + C_2 =$$

$u = \sin t$

$du = \cos t \, dt$

$\cos^2 t = 1 - u^2$

$$= \sin t - \frac{\sin^3 t}{3} + C_2$$

$$\Downarrow$$

$$y(t) = \left(\frac{\cos^3 t}{3} + C_1\right)\cos t + \left(\sin t - \frac{\sin^3 t}{3} + C_2\right)\sin t =$$

$$= \boxed{\frac{\cos^4 t}{3} + \sin^2 t - \frac{\sin^4 t}{3} + C_1 \cos t + C_2 \sin t}$$

Note that

$$\frac{\cos^4 t - \sin^4 t}{3} + \sin^2 t = \frac{\cos^2 t - \sin^2 t}{3}\underbrace{(\cos^2 t + \sin^2 t)}_{1} + \sin^2 t = \frac{\cos 2t}{3} + \frac{1-\cos 2t}{2} = \frac{1}{2} - \frac{\cos 2t}{6} \Rightarrow$$

$$y(t) = \frac{1}{2} - \frac{\cos 2t}{6} + C_1 \cos t + C_2 \sin t$$

(b) $y'' + 3y' + 2y = \dfrac{1}{1+e^t}$

Hom. eq.: $y'' + 3y' + 2y = 0$

Char. eq. $r^2 + 3r + 2 = 0 \Leftrightarrow (r+1)(r+2) = 0$

$r_1 = -1, r_2 = -2 \Rightarrow$ gen. sol. of hom. eq. is

$$y(t) = C_1 e^{-t} + C_2 e^{-2t} \Rightarrow$$

We are looking for the solutions of non-homogeneous equation in the form

$$y(t) = u_1(t) e^{-t} + u_2(t) e^{-2t} \quad \text{s.t.}$$

$$\begin{cases} e^{-t}u_1' + e^{-2t}u_2' = 0 & \text{Eq 1} \\ -e^{-t}u_1' - 2e^{-2t}u_2' = \dfrac{1}{1+e^t} & \text{Eq 2} \end{cases}$$

$\Downarrow$

Eliminating u_1' by taking (Eq 1) + (Eq 2):

$$-e^{-2t}u_2' = \frac{1}{1+e^t} \Rightarrow u_2' = -\frac{e^{2t}}{1+e^t}$$

$$\Rightarrow u_1' = -\frac{e^{-2t}u_2'}{e^{-t}} = -e^{-t}u_2' = \frac{e^t}{1+e^t}$$

$$\Rightarrow u_1 = \int \frac{e^t}{1+e^t}\,dt + \tilde{C}_1 = \int \frac{1}{u}\,du + \tilde{C}_1 = \ln|u| + C_1 =$$
$$= \ln(1+e^t) + \tilde{C}_1$$

$u = 1+e^t$

$du = e^t dt$

$$u_2 = -\int \frac{e^{2t}}{1+e^t}\,dt + \tilde{C}_2 = -\int \frac{u-1}{u}\,du + \tilde{C}_2 = -\int\left(1-\frac{1}{u}\right)du + \tilde{C}_2$$

$u = 1+e^t$

$du = e^t dt$

$$= \ln|u| - u + \tilde{C}_2 = \ln(1+e^t) - 1 - e^t + \tilde{C}_2$$
$$= \ln(1+e^t) - e^t + C_2$$

$\Downarrow$

$$y(t) = u_1(t)e^{-t} + u_2(t)e^{-2t} = e^{-t}\ln(1+e^t) + e^{-2t}\ln(1+e^t) -$$
$$-e^{-t} + \tilde{C}_1 e^{-t} + C_2 e^{-2t} = (e^{-t}+e^{-2t})\ln(1+e^t) + \underbrace{(\tilde{C}_1 - 1)}_{C_1}e^{-t} +$$
$$+ C_2 e^{-2t} = \boxed{(e^{-t}+e^{-2t})\ln(1+e^t) + C_1 e^{-t} + C_2 e^{-2t}}$$

Problem 2

a) $2\cos 4t - 5\sin 4t = R\cos(4t - \delta)$

where $R = \sqrt{2^2+5^2} = \sqrt{29}$,

$\cos\delta = \frac{2}{\sqrt{29}}$

$\sin\delta = -\frac{5}{\sqrt{29}}$ $\Rightarrow \delta$ is in the forth quadrant

$\delta =$

$\omega_0 = 4$

b) $L = 6\text{in} = \frac{1}{2}\text{ ft}$, $W = 8\text{ lb}$

The spring constant is $k = \frac{W}{L} = 16\text{ lb/ft}$

$m = \frac{W}{g} = \frac{8}{32} = \frac{1}{4}$

The equation of the spring is $mu'' + ku = 0 \Rightarrow$

$$\begin{cases} \frac{1}{4}u'' + 16u = 0 \Rightarrow u'' + 64u = 0 \\ u(0) = -2\text{in} = -\frac{1}{6}\text{ ft} \\ u'(0) = -4\text{ in/sec} = -\frac{1}{3}\text{ ft/s} \end{cases}$$

(the signs "$-$" here are because the mass is pushed up and set in motion upward)

$\Downarrow$ $r_{1,2} = \pm 8i$

$u(t) = C_1\cos 8t + C_2\sin 8t$

$u(0) = C_1 = -\frac{1}{6}$

$u'(0) = -8C_1\sin 8t + 8C_2\cos 8t\Big|_{t=0} = 8C_2 = -\frac{1}{3} \Rightarrow C_2 = -\frac{1}{24} \Rightarrow$

$u(t) = -\frac{1}{6}\cos 8t - \frac{1}{24}\sin 8t = \boxed{-\frac{1}{24}(4\cos 8t + \sin 8t)} \Rightarrow$

$R = \frac{1}{24}\sqrt{4^2+1} = \boxed{\frac{\sqrt{17}}{24}}$

$$\cos\delta = \frac{4}{\sqrt{17}} \Rightarrow \delta =$$
$$\sin\delta = \frac{1}{\sqrt{17}}$$

(c) $\gamma_{crit} = 2\sqrt{mk} = 2\sqrt{\frac{1}{4}\cdot 16} = 2\sqrt{4} = \boxed{4\ \frac{lb\cdot s}{ft}}$

Problem 3

(a) $L = 1.5\,in = \frac{1}{8}\,ft$, $W = 4\,lb \Rightarrow k = \frac{W}{L} = 32\,lb/ft$

$\gamma = 2\,\frac{lb\cdot s}{ft}$, $m = \frac{W}{g} = \frac{4}{32} = \frac{1}{8}$

In general the equation of the forced damped spring is

$$mu'' + \gamma u' + ku = F(t)$$

In our case

$$\frac{1}{8}u'' + 2u' + 32u = 3\cos 4t$$

The steady state solution is equal to the solution obtained by the method of undetermined coefficients. In our case it is of the form

$$u(t) = A_1\cos 4t + A_2\sin 4t \Rightarrow$$
$$u'(t) = 4A_2\cos 4t - 4A_1\sin 4t$$
$$u''(t) = -16A_1\cos 4t - 16A_2\sin 4t$$

$$\Rightarrow \frac{1}{8}u'' + 2u' + 32u = (-2A_1 + 8A_2 + 32A_1)\cos 4t + (-2A_2 -$$

$$-8A_1 + 32A_2)\sin 4t = (30A_1 + 8A_2)\cos 4t + (-8A_1 + 30A_2)\sin 4t =$$

$$= 3\cos 4t \Rightarrow \begin{cases} 30A_1 + 8A_2 = 3 \quad ^{\times 30} & (\text{Eq. } 1) \\ -8A_1 + 30A_2 = 0 \quad ^{\times 8} & (\text{Eq. } 2) \end{cases}$$

$$30\,\text{Eq. }1 - 8\,\text{Eq. }2 \Leftrightarrow (900 + 64)A_1 = 90 \Rightarrow$$

$$A_1 = \frac{90}{964} = \frac{45}{482}$$

$$A_2 = \frac{8}{30}A_1 =$$

$$= \frac{8}{30}\cdot\frac{45}{482} = \frac{4}{15}\cdot\frac{45}{482} = \frac{6}{241}$$

$$\Downarrow$$

$$\boxed{u(t) = \frac{45}{482}\cos 4t + \frac{6}{241}\sin 4t}$$

(b) $$1 - \frac{\gamma^2}{2km} = 1 - \frac{4}{2\cdot 32\cdot\frac{1}{8}} = 1 - \frac{4}{2\cdot 4} = \frac{1}{2} > 0$$

$$\omega_{max}^2 = \omega_0^2\left(1 - \frac{\gamma^2}{2km}\right)$$

$$\omega_0^2 = \frac{k}{m} = \frac{32}{\frac{1}{8}} = 256 \Rightarrow \omega_{max}^2 = 256\cdot\frac{1}{2} = 128 \Rightarrow$$

$$\omega_{max} = \sqrt{128} = \boxed{8\sqrt{2}}$$