Homework #8 solutions, MATH 308

Problem 1

(a) $\begin{cases} x_1' = x_1 + 2x_2 - x_3 \\ x_2' = 4x_1 - x_2 + 2x_3 \\ x_3' = 4x_1 - 4x_2 + 5x_3 \end{cases}$

Characteristic polynomial is $-\lambda^3 + 5\lambda^2 - 3\lambda - 9$

Integer roots, if exist, must be divisors of 9, i.e. they are among numbers $\pm 1, \pm 3, \pm 9$

Plug them to check if they are indeed the roots:

$\lambda = 1$: $-1 + 5 - 3 - 9 \neq 0 \to$ no

$\lambda = -1$: $1 + 5 + 3 - 9 = 0 \to$ yes $\Rightarrow$ characteristic polynomial is divisible by $\lambda + 1$:

$$\begin{array}{r|l} & \lambda^2 - 6\lambda + 9 \\ \lambda + 1 & \lambda^3 - 5\lambda^2 + 3\lambda + 9 \\ & \underline{\lambda^3 + \lambda^2} \\ & -6\lambda^2 + 3\lambda \\ & \underline{-6\lambda^2 - 6\lambda} \\ & 9\lambda + 9 \\ & \underline{9\lambda + 9} \\ & 0 \end{array}$$

$\Rightarrow \lambda^2 - 6\lambda + 9 = 0 \Leftrightarrow (\lambda - 3)^2 = 0$

$\Rightarrow$ char. polynomial is factor as

$-(\lambda + 1)(\lambda - 3)^2 \Rightarrow$

Two eigenvalues: $\boxed{\lambda = 3 \text{ of algebraic multiplicity } 2}$

and $\boxed{\lambda = -1 \text{ of algebraic multiplicity } 1}$

$\Rightarrow \boxed{\lambda = -1 \text{ has geometric multiplicity } 1}$

Find the geometric multiplicity of $\lambda = 3$

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 4 & -1 & 2 \\ 4 & -4 & 5 \end{pmatrix}$$

We want to find the dimension of the space of solutions of the system $(A-3I)v=0$ (*)

$$(A-3I)v = \begin{pmatrix} -2 & 2 & -1 \\ 4 & -4 & 2 \\ 4 & -4 & 2 \end{pmatrix} v = 0$$

$$\left(\begin{array}{ccc|c} -2 & 2 & -1 & 0 \\ 4 & -4 & 2 & 0 \\ 4 & -4 & 2 & 0 \end{array}\right) \xrightarrow[R_3 \to R_3+2R_1]{R_2 \to R_2+2R_1} \left(\begin{array}{ccc|c} -2 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \to$$

row echelon form has 2 rows of 0 $\Rightarrow$ $\boxed{\text{geometric multiplicity of } \lambda=3 \text{ is } 2}$

Geometrically system (*) is equivalent to one linear equation

$$-2v_1+2v_2+v_3=0 \quad (**)$$

which defines

i.e. again we get that the geometric multiplicity $=2$

So, in our case for every eigenvalues the geometric multiplicity = algebraic multiplicity $\Rightarrow$ A has a basis of eigenvectors $\Rightarrow$ no generalized eigenvectors (no w) here.

(b) i) Find 2 independent eigenvectors of $\lambda=3$

Referring to the equation (*) of the previous page: $-2v_1+2v_2+v_3=0$

Take $v_2=1,\ v_3=0 \Rightarrow -2v_1+2=0 \Rightarrow v_1=1$

$$v^1=\begin{pmatrix}1\\1\\0\end{pmatrix} \Rightarrow e^{3t}\begin{pmatrix}1\\1\\0\end{pmatrix} \text{ is a solution}$$

Take $v_2=0,\ v_3=1 \Rightarrow -2v_1+1=0 \Rightarrow v_1=\frac{1}{2}$

$$\Rightarrow v^2=\begin{pmatrix}\frac{1}{2}\\0\\1\end{pmatrix} \Rightarrow e^{3t} \text{ is a solution}$$

ii) Find an eigenvector of $\lambda=-1$

$$(\underbrace{A-(-I)}_{A+I})v = \begin{pmatrix}2&2&-1\\4&0&2\\4&-4&6\end{pmatrix}v=0$$

$$\left(\begin{array}{ccc|c}2&2&-1&0\\4&0&2&0\\4&-4&6&0\end{array}\right) \xrightarrow[R_3\to R_3-2R_1]{R_2\to R_2-2R_1} \left(\begin{array}{ccc|c}2&2&-1&0\\0&-4&4&0\\0&-8&8&0\end{array}\right) \xrightarrow{R_3\to R_3-2R_2}$$

$$\left(\begin{array}{ccc|c}2&2&-1&0\\0&-4&4&0\\0&0&0&0\end{array}\right) \xrightarrow{R_2\to\frac{1}{4}R_2} \left(\begin{array}{ccc|c}2&2&-1&0\\0&-1&1&0\\0&0&0&0\end{array}\right) \Rightarrow$$

$2v_1+2v_2-v_3=0$

$-v_2+v_3=0 \Rightarrow v_2=v_3$. Take $v_3=1 \Rightarrow v_2=1 \Rightarrow$

substituting to the first equation $2v_1+2-1=0 \Rightarrow v_1=-\frac{1}{2} \Rightarrow$

$V = \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 1 \end{pmatrix} \Rightarrow e^{-t}\begin{pmatrix} -\frac{1}{2} \\ 1 \\ 1 \end{pmatrix}$ is a solution

Combining all 3 found solutions we get the general solution

$$\boxed{x(t) = C_1 e^{3t}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + C_2 e^{3t}\begin{pmatrix} \frac{1}{2} \\ 0 \\ 1 \end{pmatrix} + C_2 e^{-t}\begin{pmatrix} -\frac{1}{2} \\ 1 \\ 1 \end{pmatrix}}$$

Problem 2

(a)
$$x_1' = -10x_1 - 27x_2 + 5x_3$$
$$x_2' = 7x_1 + 24x_2 - 5x_3$$
$$x_3' = -2x_1 + 18x_2 - 8x_3$$

Characteristic polynomial $= -(\lambda+3)^2(\lambda-12) \Rightarrow$

there are two eigenvalues: $\boxed{\lambda = -3 \text{ of algebraic multiplicity } 2}$

$\boxed{\lambda = 12 \text{ of algebraic multiplicity } 1} \Rightarrow$

$\boxed{\lambda = 12 \text{ has algebraic multiplicity } 1}$

Find the geometric multiplicity of $\lambda = -3$:

$$A = \begin{pmatrix} -10 & -27 & 5 \\ 7 & 24 & -5 \\ -2 & 18 & -8 \end{pmatrix}$$

We want to find the dimension of the space of solutions of the system $(A+3I)v = 0$

$$(A+3I)v = \begin{pmatrix} -7 & -27 & 5 \\ 7 & 27 & -5 \\ -2 & 18 & -5 \end{pmatrix} v = 0$$

$$\left(\begin{array}{ccc|c} -7 & -27 & 5 & 0 \\ 7 & 27 & -5 & 0 \\ -2 & 18 & -5 & 0 \end{array}\right) \xleftrightarrow{R_1 \leftrightarrow R_3} \left(\begin{array}{ccc|c} -2 & 18 & -5 & 0 \\ 7 & 27 & -5 & 0 \\ -7 & -27 & 5 & 0 \end{array}\right) \xrightarrow[R_2 \to 2R_2 + 7R_1]{R_3 \to R_3 + R_2} \left(\begin{array}{ccc|c} -2 & 18 & -5 & 0 \\ 0 & \underbrace{180}_{54+126} & -45 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_2 \to \frac{1}{45}R_2} \left(\begin{array}{ccc|c} -2 & 18 & -5 & 0 \\ 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \to$$ row echelon form has 1 row of zeros $\Rightarrow$ geometric multiplicity of $\lambda = 3$ is 1

Geometrically system $(A+3I)v = 0$ is equivalent to a system of 2 linearly independent equation.

$$\begin{cases} -2v_1 + 18v_2 - 5v_3 = 0 \\ 4v_2 - v_3 = 0 \end{cases} \to$$ intersection of 2 different planes passing through the origin $\Rightarrow$ a line $\Rightarrow$ geometric multiplicity of $\lambda = 3$ is 1

(b) (Following algorithm 2)

i) Find an eigenvector of $\lambda = 3$

$$\begin{cases} -2v_1 + 18v_2 - 5v_3 = 0 & (Eq\ 1) \\ 4v_2 - v_3 = 0 & (Eq\ 2) \end{cases}$$

Take $v_2 = 1 \overset{(Eq\ 2)}{\Longrightarrow} v_3 = 4 \overset{(Eq\ 1)}{\Longrightarrow} -2v_1 + 18 - 20 = 0 \Rightarrow v_1 = -1$

$$\Rightarrow v = \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix}$$

ii) Find w s.t $(A+3I)w = v$ from the previous item

$$\left(\begin{array}{ccc|c} 7 & -27 & 5 & -1 \\ 7 & 27 & -5 & 1 \\ -2 & 18 & -5 & 4 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_3} \left(\begin{array}{ccc|c} -2 & 18 & -5 & 4 \\ 7 & 27 & -5 & 1 \\ -7 & -27 & 5 & -1 \end{array}\right) \xrightarrow[R_2 \to 2R_2+7R_1]{R_2 \to R_2+R_3}$$

$$\left(\begin{array}{ccc|c} -2 & 18 & -5 & 4 \\ 0 & 180 & -45 & 30 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_2 \to \frac{1}{15}R_2} \left(\begin{array}{ccc|c} -2 & 18 & -5 & 4 \\ 0 & 12 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$-2w_1 + 18w_2 - 5w_3 = 4 \quad (\text{Eq } 3)$$
$$12w_2 - 3w_3 = 2 \quad (\text{Eq } 4)$$

Take $w_3 = 0 \Rightarrow 12w_2 = 2 \overset{(\text{Eq }4)}{\Rightarrow} w_2 = \frac{1}{6} \overset{(\text{Eq }3)}{\Rightarrow} -2w_1 + 3 = 4 \Rightarrow w_1 = -\frac{1}{2}$

So $w = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{6} \\ 0 \end{pmatrix} \Rightarrow$ $e^{-3t}v = e^{-3t}\begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix}$ and

$e^{-3t}(w+tv) = e^{-3t}\left(\begin{pmatrix} -\frac{1}{2} \\ \frac{1}{6} \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix}\right)$

are independent solution

iii) Find an eigenvector of $\lambda = 12$:

$$(A-12I)v = \begin{pmatrix} -22 & -27 & 5 \\ 7 & 12 & -5 \\ -2 & 18 & -20 \end{pmatrix} v = 0$$

$$\left(\begin{array}{ccc|c} -22 & -27 & 5 & 0 \\ 7 & 12 & -5 & 0 \\ -2 & 18 & -20 & 0 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_3} \left(\begin{array}{ccc|c} -2 & 18 & -20 & 0 \\ 7 & 12 & -5 & 0 \\ -22 & -27 & 5 & 0 \end{array}\right) \xrightarrow{R_1 \to -\frac{1}{2}R_1}$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & -9 & 10 & 0 \\ 7 & 12 & -5 & 0 \\ -22 & -27 & 5 & 0 \end{array}\right) \xrightarrow[R_3 \to R_3 + 22R_1]{R_2 \to R_2 - 7R_1} \left(\begin{array}{cccc} 1 & -9 & 10 & 0 \\ 0 & 75 & -75 & 0 \\ 0 & -225 & 225 & 0 \end{array}\right) \longrightarrow$$

$-27 - 22 \cdot 9 = -225$

$$\xrightarrow[R_3 \to \frac{1}{225} R_3]{R_2 \to \frac{1}{75} R_2} \left(\begin{array}{ccc|c} 1 & -9 & 10 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right) \xrightarrow{R_3 \to R_3 - R_2} \left(\begin{array}{ccc|c} 1 & -9 & 10 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow$ $v_1 - 9v_2 + 10v_3 = 0$

$v_2 - v_3 = 0$ $\Rightarrow$ take $v_3 = 1 \Rightarrow v_2 = 1 \Rightarrow$

Substituting to the first equation

$v_1 - 9 + 10 = 0 \Rightarrow v_1 = -1 \Rightarrow$

$\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector of $\lambda = 12 \Rightarrow e^{12t}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ is

a solution

Combining all we have that the general solution is

$$\boxed{\begin{aligned} x(t) &= c_1 e^{-3t}\begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix} + c_2 e^{-3t}\left(\begin{pmatrix} -\frac{1}{2} \\ \frac{1}{6} \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix}\right) + c_3 e^{12t}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \\ &e^{-3}\begin{pmatrix} -c_1 - \frac{1}{2}c_2 - c_2 t \\ c_1 + \frac{1}{6}c_2 + c_2 t \\ 4c_1 + 4c_2 t \end{pmatrix} + c_3 e^{12t}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \end{aligned}}$$

Solution of Problem 2 (b) using algorithm 1

We found that the eigenline of $\lambda = -3$ is generated by $\begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix}$ (see item 8 i) of page 5)

Find the space $E_3^{(2)}$ of all generalized eigenvectors of order ≤ 2 (which in this case coinsides with the space of all eigenvectors) by finding the space of solutions of the equation $(A+3I)^2 w = 0$

$$(A+3I)^2 = \begin{pmatrix} -7 & -27 & 5 \\ 7 & 27 & -5 \\ -2 & 18 & -5 \end{pmatrix} \begin{pmatrix} -7 & -27 & 5 \\ 7 & 27 & -5 \\ -2 & 18 & -5 \end{pmatrix} =$$

$$= \begin{pmatrix} 49-189-10 & (7-27)27+90 & (-7+27)5-25 \\ (-7+27)\cdot 7+10 & (-7+27)\cdot 27-90 & (7-27)\cdot 5+25 \\ (2+18)\cdot 7+10 & (2+18)\cdot 27-90 & (-2-18)\cdot 5+25 \end{pmatrix} =$$

$$= \begin{pmatrix} -150 & -450 & 75 \\ 150 & 450 & -75 \\ 150 & 450 & -75 \end{pmatrix}$$

Rem: By the theory the resulting matrix had to have all rows being multiple to one of the rows

$\Rightarrow (A+3I)^2 w = 0 \Leftrightarrow 150w_1 + 450w_2 - 75w_3 = 0 \Leftrightarrow$

$2w_1 + 6w_2 - w_3 = 0 \quad (***)$

Choose any w satisfying $(***)$ which is not an eigenvector of $\lambda=-3$, i.e not collinear to $\begin{pmatrix}-1\\1\\4\end{pmatrix}$

For exemple the vector $w = \begin{pmatrix}-\frac{1}{2}\\ \frac{1}{6}\\ 0\end{pmatrix}$ chosen when

we implemented Algorithm 2 will work and the rest

of the solution will be the same (of course as w

you can take other vectors in both algorithms)

<u>Problem 3</u> a) $\begin{cases} x_1' = 4x_1 + 12x_2 + 2x_3 \\ x_2' = -5x_1 - 12x_2 - x_3 \\ x_3' = -20x_1 - 24x_2 - 10x_3 \end{cases}$

If the characteristic polynomial is $-(\lambda+6)^3$ then

there is only one eigenvalue $\boxed{\lambda=-6 \text{ of algebraic multiplicity } 3}$

Find its geometric multiplicity by finding the dimension

of the space of solution of the system $(A+6I)v=0$

$$(A+6I)v = \begin{pmatrix} 10 & 12 & 2 \\ -5 & -6 & -1 \\ -20 & -24 & -4 \end{pmatrix} v = 0$$

$$\left(\begin{array}{ccc|c} 10 & 12 & 2 & 0 \\ -5 & -6 & -1 & 0 \\ -20 & -24 & -4 & 0 \end{array}\right) \xrightarrow[R_2 \to -R_2,\ R_3 \to -\frac{R_3}{4}]{R_1 \to \frac{R_1}{2}} \left(\begin{array}{ccc|c} 5 & 6 & 1 & 0 \\ 5 & 6 & 1 & 0 \\ 5 & 6 & 1 & 0 \end{array}\right) \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \left(\begin{array}{ccc|c} 5 & 6 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow$ row echelon form has 2 rows of zeros $\Rightarrow$ $\boxed{\text{geometric multiplicity of } \lambda=-6 \text{ is } 2}$

 In fact the eigenspace of $\lambda=-6$

is given by equation

$$5v_1+6v_2-v_3=0 \to (\text{a plane}) \qquad (****)$$

b) We use the analog of algorithm 1:

Take any w which is not in the eigenspace of $\lambda=-6$ i.e. does not satisfy equation $(****)$

For example take $w=\begin{pmatrix}1\\0\\0\end{pmatrix}$

Let $v^1=(A+6I)w=\begin{pmatrix}10\\-5\\20\end{pmatrix}$ (note that v^1 satisfies $(****)$)

Take an eigenvector v^2 which is not collinear to v_1

For this plug for example $v_1=1$, $v_2=0$ in $(****)$

Then $5-v_3=0 \Rightarrow v_3=5 \Rightarrow$ we can take

$$v^2=\begin{pmatrix}1\\0\\5\end{pmatrix}$$

So $\{v^1,w,v^2\}$ is a basis of generalized eigenvectors

$$e^{tA}v^1=e^{-6t}v^1=e^{-6t}\begin{pmatrix}10\\-5\\20\end{pmatrix}$$

$$e^{tA}w=e^{-6t}(w+tv^1)=e^{-6t}\left(\begin{pmatrix}1\\0\\0\end{pmatrix}+t\begin{pmatrix}10\\-5\\20\end{pmatrix}\right)$$

$$e^{tA}v^2=e^{-6t}v^2=e^{-6t}\begin{pmatrix}1\\0\\5\end{pmatrix}$$

So, the general solution is

$$\boxed{x(t) = e^{-6t}\left(C_1 \begin{pmatrix} 10 \\ -5 \\ 20 \end{pmatrix} + C_2\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 10 \\ -5 \\ 20 \end{pmatrix}\right) + C_3 \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix}\right) = e^{-6t}\begin{pmatrix} 10C_1 + C_2 + C_3 + 10C_2 t \\ -5C_1 - 5C_2 t \\ 20C_1 + 5C_3 + 20C_2 t \end{pmatrix}}$$

Rem 1 Algorithm 2 will not work pretty well here and essentially it reduces to algorithm 2

The point is that not for any v from the eigenplane of $\lambda = -6$ there exist w s.t. $(A+6I)w = v$

In fact $(A+6I)w = \begin{pmatrix} 10w_1 + 12w_2 - 2w_3 \\ -5w_1 - 6w_2 - w_3 \\ -20w_1 - 24w_2 - 4w_3 \end{pmatrix} = (5w_1 + 6w_2 - w_3)\begin{pmatrix} -2 \\ 1 \\ -4 \end{pmatrix}$

So the eq. $(A+6I)w = v$ will have a solution $\Leftrightarrow$ v is collinear to $\begin{pmatrix} -2 \\ 1 \\ -4 \end{pmatrix}$

(note that the last vector satisfies (****) of page 10). We can take $v = \begin{pmatrix} -2 \\ 1 \\ -4 \end{pmatrix}$ or any its multiple and find w from $(A+6I)w = v$

But it is easier to choose any w which is not an eigenvector and to proceed as in algorithm 1

Rem2 As mentioned on page 1 of notes 27 in the considered case $(A+6I)^2 = 0$

 Indeed

$$(A+6I)^2 = \begin{pmatrix} 10 & 12 & 2 \\ -5 & -6 & -1 \\ -20 & -24 & -4 \end{pmatrix}\begin{pmatrix} 10 & 12 & 2 \\ -5 & -6 & -1 \\ -20 & -24 & -4 \end{pmatrix} = \begin{pmatrix} 100-60-40 & 120-72-48 & 20-12-8 \\ -50+30+20 & 60-36-24 & -10+6+4 \\ -200+120+80 & -240+144+96 & -40+24+16 \end{pmatrix} = 0$$

(you do not have to verify it, it follows from the general theory)

So you can proceed as in algorithm 3 without finding eigenvectors and generalized eigenvectors

$$e^{At} = e^{-6t} e^{(A+6I)t} = e^{-6t}\left(I + (A+6I)t \right) =$$

$$= e^{-6t}\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + t \begin{pmatrix} 10 & 12 & 2 \\ -5 & -6 & -1 \\ -20 & -24 & -4 \end{pmatrix} \right) =$$

$$= e^{-6t}\begin{pmatrix} 1+10t & 12t & 2t \\ -5t & 1-6t & -t \\ -20t & -24t & 1-4t \end{pmatrix} \Rightarrow$$ the columns

of this matrix form a fundamental set of solutions $\Rightarrow$ The general solution can be written in the form

$$\boxed{x(t) = e^{-6t}\left(c_1 \begin{pmatrix} 1+10t \\ -5t \\ -20t \end{pmatrix} + c_2 \begin{pmatrix} 12t \\ 1-6t \\ -24t \end{pmatrix} + c_3 \begin{pmatrix} 2t \\ -t \\ 1-4t \end{pmatrix} \right)}$$

Problem 4 a)

$$\begin{cases} x_1' = 2x_1 + 2x_2 + x_3 \\ x_2' = -\frac{3}{2}x_1 - 2x_2 - \frac{1}{2}x_3 \\ x_3' = -8x_1 - 8x_2 - 3x_3 \end{cases}$$

Page 13) Since the characteristic polynomial is $-(\lambda+1)^3$

there is one eigenvalue $\boxed{\lambda = -1 \text{ of algebraic multiplicity } 3}$

Find its geometric multiplicity by finding the dimension of the eigenspace, i.e. the dimension of space of solutions of the system $(A+I)v=0$

$$(A+I)v = \begin{pmatrix} 3 & 2 & 1 \\ -\frac{3}{2} & -1 & -\frac{1}{2} \\ -8 & -8 & -2 \end{pmatrix} v = 0$$

$$\left(\begin{array}{ccc|c} 3 & 2 & 1 & 0 \\ -\frac{3}{2} & -1 & -\frac{1}{2} & 0 \\ -8 & -8 & -2 & 0 \end{array}\right) \xrightarrow[R_3 = 3R_3 + 8R_1]{R_2 \to R_2 + \frac{1}{2}R_1} \left(\begin{array}{ccc|c} 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -8 & 2 & 0 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 3 & 2 & 1 & 0 \\ 0 & -8 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \to$$

$$\xrightarrow{R_2 \to -\frac{1}{2}R_2} \left(\begin{array}{ccc|c} 3 & 2 & 1 & 0 \\ 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The row echelon form has one zero row $\Rightarrow$ $\boxed{\text{geometric multiplicity of } \lambda=-1 \text{ is equal to } 1}$

b) Proceed with analog of algorithm 2

i) Find an eigenvector of $\lambda = -1$

From the calculations in the previous item

$$3v_1 + 2v_2 + v_3 = 0$$
$$4v_2 - v_3 = 0 \Rightarrow \text{ take } v_2 = 1 \Rightarrow v_3 = 4 \Rightarrow \text{substi-}$$

tuting to the 1st equation: $3v_1 + 2 + 4 = 0 \Rightarrow v_1 = -2$

$$\Rightarrow v = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$$

ii) Find w^1 s.t $(A+I)w^1 = \underline{v}$

as in the previous item

$$\left(\begin{array}{ccc|c} 3 & 2 & 1 & -2 \\ -\frac{3}{2} & -1 & -\frac{1}{2} & 1 \\ -8 & -8 & -2 & 4 \end{array}\right) \xrightarrow[R_3 \to 3R_3 + 8R_1]{R_2 \to R_2 + \frac{1}{2}R_1} \left(\begin{array}{ccc|c} 3 & 2 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & -8 & 2 & -4 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_3}$$

$$\left(\begin{array}{ccc|c} 3 & 2 & 1 & -2 \\ 0 & -8 & 2 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_2 \to -\frac{1}{2}R_2} \left(\begin{array}{ccc|c} 3 & 2 & 1 & -2 \\ 0 & 4 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$3w_1 + 2w_2 + w_3 = -2 \quad \text{(Eq 1)}$$
$$4w_2 - w_3 = 2 \quad \text{(Eq 2)}$$

Take $w_3 = 0 \overset{(2)}{\Rightarrow} w_2 = \frac{1}{2} \overset{\text{(Eq 1)}}{\Rightarrow} 3w_1 + 1 = -2 \Rightarrow w_1 = -1$

$$\Rightarrow w^1 = \begin{pmatrix} -1 \\ \frac{1}{2} \\ 0 \end{pmatrix}$$

iii) Find w^2 s.t $(A+I)w^2 = \underline{w^1}$

found in the previous item

$$\left(\begin{array}{ccc|c} 3 & 2 & 1 & -1 \\ -\frac{3}{2} & -1 & -\frac{1}{2} & \frac{1}{2} \\ -8 & -8 & -2 & 0 \end{array}\right) \xrightarrow[R_3 \to 3R_3 + 8R_1]{R_2 \to R_2 + \frac{1}{2}R_1} \left(\begin{array}{ccc|c} 3 & 2 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -8 & 2 & -8 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_3}$$

$$\left(\begin{array}{ccc|c} 3 & 2 & 1 & -1 \\ 0 & -8 & 2 & -8 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_2 \to -\frac{1}{2}R_2} \left(\begin{array}{ccc|c} 3 & 2 & 1 & -1 \\ 0 & 4 & -1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$3w_1 + 2w_2 + w_3 = -1$$
$$4w_2 - w_3 = 4$$

Take $w_3 = 0 \Rightarrow w_2 = 1 \Rightarrow$
$3w_1 + 2 = -1 \Rightarrow w_1 = -1 \Rightarrow w^2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

So, $\{v, w^1, w^2\}$ is a basis of generalized eigenvectors

$$e^{tA}v = e^{-t}v = e^{-t}\begin{pmatrix}-2\\1\\4\end{pmatrix}$$

$$e^{tA}w^1 = e^{-t}(w^1+tv) = e^{-t}\left(\begin{pmatrix}-1\\ \frac{1}{2}\\0\end{pmatrix}+t\begin{pmatrix}-2\\1\\4\end{pmatrix}\right)$$

$$e^{tA}w^2 = e^{-t}\left(w^2+tw^1+\frac{t^2}{2}v\right) = e^{-t}\left(\begin{pmatrix}-1\\1\\0\end{pmatrix}+t\begin{pmatrix}-1\\ \frac{1}{2}\\0\end{pmatrix}+\frac{t^2}{2}\begin{pmatrix}-2\\1\\4\end{pmatrix}\right)$$

So the general solution is

$$\boxed{\begin{aligned} X(t) = e^{-t}\Bigg(& c_1\begin{pmatrix}-2\\1\\4\end{pmatrix} + c_2\left(\begin{pmatrix}-1\\ \frac{1}{2}\\0\end{pmatrix}+t\begin{pmatrix}-2\\1\\4\end{pmatrix}\right) + \\ & + c_3\left(\begin{pmatrix}-1\\1\\0\end{pmatrix}+t\begin{pmatrix}-1\\ \frac{1}{2}\\0\end{pmatrix}+\frac{t^2}{2}\begin{pmatrix}-2\\1\\4\end{pmatrix}\right)\Bigg)\end{aligned}}$$

- Problem 4 b using algorithm 1 (end of page 13)

i) As in item i) of algorithm 2 we find that the eigenline of $\lambda=-1$ is generated by $\begin{pmatrix}-2\\1\\4\end{pmatrix}$

ii) Find the space $E_{-1}^{(2)}$ of generalized eigenvectors of order ≤ 2 by finding solutions of $(A+I)^2 w = 0$

$$(A+I)^2 = \begin{pmatrix}3 & 2 & 1\\ -\frac{3}{2} & -1 & -\frac{1}{2}\\ -8 & -8 & -2\end{pmatrix}\begin{pmatrix}3 & 2 & 1\\ -\frac{3}{2} & -1 & -\frac{1}{2}\\ -8 & -8 & -2\end{pmatrix} = \begin{pmatrix}9-3-8 & 6-2-8 & 3-1-2\\ -\frac{9}{2}+\frac{3}{2}+4 & -3+1+4 & -\frac{3}{2}+\frac{1}{2}+1\\ -24+12+16 & -16+8+16 & -8+4+4\end{pmatrix} =$$

$$= \begin{pmatrix} -2 & -4 & 0 \\ 1 & 2 & 0 \\ 4 & 8 & 0 \end{pmatrix}$$

(by general theory in this matrix all rows must be a multiple of one of the rows)

$\Rightarrow \quad (A+I)^2 w = 0 \quad \Leftrightarrow \quad w_1 + 2w_2 = 0$

Take eigvector w^2 which is not in $E_{-1}^{(2)}$

For example we can take $w^2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ as in the solution

using algorithm 2 and then set $w^1 = (A+I)w^2$,

$v = (A+I)w^1 = (A+I)^2 w^2$

and proceed further as in algorithm 2

We can take $w^2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, then $w^1 = (A+I)w^2 = \begin{pmatrix} 3 \\ -\frac{3}{2} \\ -8 \end{pmatrix}$

$v = (A+I)w^1 = (A+I)^2 w^2 = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$ and then we can

proceed as in algorithm 2 with this triple of $\{v, w^1, w^2\}$.

- Problem 46 using algorithm 3

As mentioned on page 1 of notes 27 in the considered case

$(A+I)^3 = 0$ (this is actually a consequence of Cayley-Hamilton Thm)

(Indeed $(A+I)^3 = (A+I)^2(A+I) = \begin{pmatrix} -2 & -4 & 0 \\ 1 & 2 & 0 \\ 4 & 8 & 0 \end{pmatrix} \begin{pmatrix} 3 & 2 & 1 \\ -\frac{3}{2} & -1 & -\frac{1}{2} \\ -8 & -8 & -2 \end{pmatrix} = 0$

So we can find e^{At} without finding eigenvectors and generalized eigenvectors

$$e^{At} = e^{-t} e^{(A+I)t} = e^{-t}\left(I + t(A+I) + \frac{t^2}{2}(A+I)^2 \right) =$$

$$= e^{-t}\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + t \begin{pmatrix} 3 & 2 & 1 \\ -\frac{3}{2} & -1 & -\frac{1}{2} \\ -8 & -8 & -2 \end{pmatrix} + \frac{t^2}{2} \begin{pmatrix} -2 & -4 & 0 \\ 1 & 2 & 0 \\ 4 & 8 & 0 \end{pmatrix} \right) =$$

$$= e^{-t} \begin{pmatrix} 1+3t-t^2 & 2t-2t^2 & t \\ -\frac{3}{2}t+\frac{1}{2}t^2 & 1-t+t^2 & -\frac{1}{2}t \\ -8t+2t^2 & -8t+4t^2 & 1-2t \end{pmatrix}$$

$\Rightarrow$ the columns of this matrix form a fundamental set of solutions $\Rightarrow$ general solution is

$$\boxed{x(t) = e^{-t}\left(c_1 \begin{pmatrix} 1+3t-t^2 \\ -\frac{3}{2}t+\frac{1}{2}t^2 \\ -8t+2t^2 \end{pmatrix} + c_2 \begin{pmatrix} 2t-2t^2 \\ 1-t+t^2 \\ -8t+4t^2 \end{pmatrix} + c_3 \begin{pmatrix} t \\ -\frac{1}{2}t \\ 1-2t \end{pmatrix} \right)}$$

$\rightarrow$ an equivalent form of the answer to one on page 15