Homework assignment #8 - Solution - MATH 308 - Summer 2012

Problem 1 a) $C_1 = -3,\ C_2 = 4 \Rightarrow R = \sqrt{C_1^2+C_2^2} = \sqrt{9+16} = \boxed{5}$

$\boxed{\omega_0 = 6}$ , $\cos\delta = -\frac{3}{5}$, $\sin\delta = \frac{4}{5}$ $\Rightarrow$ $\delta$ is in the second quadrant

$\Rightarrow \delta = \pi - \arccos\frac{3}{5} \approx 2.214\text{ rad} \approx 126.87°$

b) $W = 16lb$, $L = 1.5in = \frac{1}{8}ft \Rightarrow$

The spring constant $k$ satisfies $k = \frac{W}{L} = \frac{16lb}{\frac{1}{8}ft} = 128\frac{lb}{ft}$;

$m = \frac{W}{g} = \frac{16}{32} = \frac{1}{2}$

The equation of the spring is $mu'' + ku = 0 \Leftrightarrow$

$$\begin{cases} \frac{1}{2}u'' + 128u = 0 \Leftrightarrow u'' + 256u = 0 \\ u(0) = 4in = \frac{1}{3}ft \\ u'(0) = -1ft/s \quad \text{(because it is set in motion upward)} \end{cases}$$

The characteristic equation is $r^2 + 256 = 0 \Rightarrow r_{1,2} = \pm 16i \Rightarrow$

$\lambda = 0, \mu = 8 \Rightarrow$ the natural frequency is $\boxed{\omega_0 = 16}$ $\Rightarrow$ the period $\boxed{T = \frac{2\pi}{16} = \frac{\pi}{8}}$

$u(t) = C_1\cos 16t + C_2\sin 16t$

$u(0) = \frac{1}{3} \Rightarrow C_1 = \frac{1}{3}$

$u'(t) = -16\sin 16t + 16C_2\cos 16t \Rightarrow$

$u'(0) = -1 \Leftrightarrow 16C_2 = -1 \Rightarrow C_2 = -\frac{1}{16}$

$u(t) = \frac{1}{3}\cos 16t - \frac{1}{16}\sin 16t$

$R = \sqrt{\frac{1}{9}+\frac{1}{256}} = \frac{\sqrt{265}}{3\cdot 16} = \frac{\sqrt{265}}{48} \Rightarrow$

$\cos\delta = \frac{\frac{1}{3}}{\frac{\sqrt{265}}{48}} = \frac{16}{\sqrt{265}}$, $\sin\delta = \frac{-\frac{1}{16}}{\frac{\sqrt{265}}{48}} = -\frac{3}{\sqrt{265}} \Rightarrow$

$\Rightarrow \delta$ is in the forth quadrant $\Rightarrow$

$$\boxed{\delta = 2\pi - \arccos\frac{16}{\sqrt{265}} \approx 6.097 \text{ rad} \approx 349.38^\circ \quad (\text{or} \approx -10.62^\circ)}$$

c) $\gamma = 2\sqrt{mk} = 2\sqrt{\frac{1}{2}\cdot 128} = 2\sqrt{64} = 2\cdot 8 = \boxed{16 \ \frac{lb\cdot s}{ft}}$

Problem 2 (a) The characteristic equation is

$r^2 - 6r + 9 = 0 \Leftrightarrow (r-3)^2 = 0 \Rightarrow r_{1,2} = 3 \Rightarrow$

gen. solution is $\boxed{y(t) = C_1 e^{3t} + C_2 t e^{3t}}$

(b) $y(0) = \alpha \Leftrightarrow \boxed{C_1 = \alpha}$

$y'(t) = 3C_1 e^{3t} + C_2 e^{3t} + 3C_2 t e^{3t}$

$y'(0) = 3C_1 + C_2 = -2 \Rightarrow 3\alpha + C_2 = -2 \Rightarrow \boxed{C_2 = -2 - 3\alpha}$

$\Rightarrow \boxed{y(t) = \alpha e^{3t} - (3\alpha+2) t e^{3t}} = (\alpha - (3\alpha+2)t) e^{3t}$

c) $y(t) \xrightarrow[t\to+\infty]{} +\infty \Leftrightarrow 3\alpha - 2 \leq 0 \Leftrightarrow \boxed{\alpha < -\frac{2}{3}}$

$y(t) \xrightarrow[t\to+\infty]{} -\infty \Leftrightarrow 3\alpha + 2 \leq 0 \Leftrightarrow \boxed{\alpha > -\frac{2}{3}}$