Solutions of Homework #9 MATH 32?

Problem 1

$$\begin{cases} x_1' = x_1 - 16x_2 \\ x_2' = x_1 + 9x_2 \end{cases}$$

(a) $A = \begin{pmatrix} 1 & -16 \\ 1 & 9 \end{pmatrix}$

- $\det(A - \lambda I) = \lambda^2 - \operatorname{tr} A\, \lambda + \det A = \lambda^2 - 10\lambda + 25 = (\lambda - 5)^2$

  $\operatorname{tr} A = 10$, $\det A = 1 \cdot 9 - 1 \cdot (-16) = 25$

$\Rightarrow \lambda_{1,2} = 5 \Rightarrow$ the algebraic multiplicity of $\lambda = 5$ is 2

- Geometric multiplicity of $\lambda = 5$?

  $(A - 5I)v = \begin{pmatrix} -4 & -16 \\ 1 & 4 \end{pmatrix} v = 0 \iff v_1 + 4v_2 = 0$

$\Rightarrow v_1 = -4v_2 \Rightarrow$ the eigenspace $E_5 = \left\{ v_2 \begin{pmatrix} -4 \\ 1 \end{pmatrix}, v_2 \in \mathbb{R} \right\}$

$\Rightarrow \dim E_5 = 1$ (generated by one vector $\begin{pmatrix} -4 \\ 1 \end{pmatrix}$) $\Rightarrow$

the geometric multiplicity of $\lambda = 5$ is 1

- Now we proceed in 3 different ways according 3 algorithm I presented in class

1) Using algorithm 1

- Take any $w \in \mathbb{R}^2$ s.t $w \notin E_\lambda$, for example,

  $w = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

- Let $v = (A-5I)w = \begin{pmatrix} -4 & -16 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$

Then (note that automatically v is an eigen vector, i.e. belongs to E_5)

- $e^{At}v, e^{At}w$ is a fundamental set of solution

$$e^{At}v = e^{\lambda t}v = e^{5t}\begin{pmatrix} -4 \\ 1 \end{pmatrix}$$

$$e^{At}w = e^{\lambda t}(w+tv) = e^{5t}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -4 \\ 1 \end{pmatrix}\right)$$

$\Rightarrow$ the general solution is

$$\boxed{X(t) = e^{5t}\left(C_1\begin{pmatrix} -4 \\ 1 \end{pmatrix} + C_2\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -4 \\ 1 \end{pmatrix}\right)\right)}$$

2) Using algorithm 2

- Pick up some v in E_5, for example

$$v = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$$

- Find w s.t $(A-\lambda I)w = v$

$$\begin{pmatrix} -4 & -16 \\ 1 & 4 \end{pmatrix} w = \begin{pmatrix} -4 \\ 1 \end{pmatrix} \Rightarrow w_1 + 4w_2 = 1$$

let $w_2 = 0 \Rightarrow w_1 = 1 \Rightarrow w = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow$ the rest is the same as in algorithm 1.

3) <u>Using algorithm 3</u> For any w: $(A-5I)^2 w = 0 \Leftrightarrow (A-5I)^2 = 0$

$$e^{At} = e^{5t} e^{(A-5I)t} \underset{\substack{\uparrow \\ \text{Because } (A-5I)^2 = 0}}{=} e^{5t}\left(I + (A-5I)t\right) =$$

$$= e^{5t}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t\begin{pmatrix} -4 & -16 \\ 1 & 4 \end{pmatrix}\right) = \underbrace{e^{5t}\begin{pmatrix} 1-4t & -16t \\ t & 1+4t \end{pmatrix}}_{\text{fundamental matrix}}$$

$\Rightarrow$ the columns of this matrix form a fundamental set of solutions $\Rightarrow$ the general solution is

$$x(t) = e^{5t}\left(\tilde{c}_1 \begin{pmatrix} 1-4t \\ t \end{pmatrix} + \tilde{c}_2 \begin{pmatrix} -16t \\ 1+4t \end{pmatrix}\right)$$

<u>Rem</u> It gives the same set of solutions as in the previous methods: let us find the relation between constants (c_1, c_2) and $(\tilde{c}_1, \tilde{c}_2)$

Comparing coeff. of t: $c_2 = \tilde{c}_1 + 4\tilde{c}_2$ $(*)$

Comparing free term: $c_1\begin{pmatrix} -4 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \tilde{c}_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \tilde{c}_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\Rightarrow \begin{array}{l} \tilde{c}_1 = -4c_1 + c_2 \\ \tilde{c}_2 = c_1 \end{array}$$ $\rightarrow$ this is compatible with $(*)$

b) $e^{5t} \underset{t \to -\infty}{\longrightarrow} 0 \Rightarrow x(t) \underset{t \to -\infty}{\longrightarrow} 0$ independently of initial conditions

(c) $\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} -4 \\ 5 \end{pmatrix} \underset{\uparrow}{=} c_1 \begin{pmatrix} -4 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow$

using
solution of alg. 1&2

$-4C_1 + C_2 = -4 \qquad -20 + C_2 = -4 \Rightarrow \boxed{C_2 = 16}$

$\boxed{C_1 = 5}$

$\Rightarrow X(t) = e^{5t} \left(5 \begin{pmatrix} -4 \\ 1 \end{pmatrix} + 16 \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -4 \\ 1 \end{pmatrix} \right) \right) =$

$= \boxed{e^{5t} \left(\begin{pmatrix} -4 \\ 5 \end{pmatrix} + t \begin{pmatrix} -64 \\ 16 \end{pmatrix} \right)}$

* Note that the solution using algorithm 3 gives us the solution of IVP immediately (without solving an algebraic systems)

$X(t) = e^{At} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = e^{5t} \begin{pmatrix} 1-4t & -16t \\ t & 1+4t \end{pmatrix} \begin{pmatrix} -4 \\ 5 \end{pmatrix} =$

$= e^{-5t} \begin{pmatrix} -4 + 16t - 80t \\ -4t + 5 + 20t \end{pmatrix} = e^{-5t} \begin{pmatrix} -4 - 64t \\ 5 + 16t \end{pmatrix}$

(the same as before)

Problem 2

$$\begin{cases} x_1' = -16x_1 - 3x_2 - 21x_3 \\ x_2' = 12x_1 - x_2 + 21x_3 \\ x_3' = 12x_1 + 3x_2 + 17x_3 \end{cases} \qquad A = \begin{pmatrix} -16 & -3 & -21 \\ 12 & -1 & 21 \\ 12 & 3 & 17 \end{pmatrix}$$

(a) $\lambda_1 = -4$ is an eigenvalue $\Rightarrow$

the characteristic polynomial $-\lambda^3 + 48\lambda + 128$

is divisible by $\lambda + 4$:

$$\begin{array}{r|l} & -\lambda^2 + 4\lambda + 32 \\ \lambda + 4 & -\lambda^3 + \quad 48\lambda + 128 \\ & -\lambda^3 - 4\lambda^2 \\ \hline & \quad 4\lambda^2 + 48\lambda \\ & \quad 4\lambda^2 + 16\lambda \\ \hline & \qquad 32\lambda + 128 \\ & \qquad 32\lambda + 128 \\ \hline & \qquad\qquad 0 \end{array} \Rightarrow$$

$-\lambda^2 + 4\lambda + 32 = 0 \Leftrightarrow$

$\lambda^2 - 4\lambda - 32 = 0$

$D = 16 + 128 = 144$

$\lambda_1 = \frac{4+12}{2} = 8$

$\lambda_2 = \frac{4-12}{4} = -4$ $\Rightarrow$

So the eigenvalues are $\lambda_1 = -4$ and $\lambda_2 = 8$

$\lambda_1 = -4$ has the algebraic multiplicity 2

$\lambda_1 = 8$ has the algebraic multiplicity 1 $\Rightarrow$ geometric multiplicity must be 1

Find the geometric multiplicity of $\lambda_1 = -4$

$$\left(A-(-4I)\right)v = (A+4I)v \Rightarrow \begin{pmatrix} -12 & -3 & -21 \\ 12 & 3 & 21 \\ 12 & 3 & 21 \end{pmatrix} v = 0$$

$$\left(\begin{array}{ccc|c} -12 & -3 & -21 & 0 \\ 12 & 3 & 21 & 0 \\ 12 & 3 & 21 & 0 \end{array}\right) \underset{\substack{R_3 \to R_3 + R_1 \\ R_1 \Rightarrow -\frac{1}{3}R_1}}{\overset{R_2 \to R_2 + R_1}{\sim}} \left(\begin{array}{ccc|c} 4 & 1 & 7 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \text{geom. multiplicity is } \underline{2}$$

The eigenspace E_{-4} is the plane satisfying

$$4v_1 + v_2 + 7v_3 = 0 \qquad (*)$$

(b) i) Since geom. multiplicity of $\lambda_1 = -4$ = alg. multiplicity of $\lambda_1 = -4$ (and equal to 2) we deal with eigenvectors only (no w here!)

Choose a basis in the eigenspace E_{-4}:

1) Take $v_2 = 1, v_3 = 0 \overset{(*)}{\Rightarrow}$ (plug into $(*)$)

$$4v_1 + 1 = 0 \Rightarrow v_1 = -\frac{1}{4} \Rightarrow \text{one eigenvector can}$$

be taken as $v^1 = \begin{pmatrix} -\frac{1}{4} \\ 1 \\ 0 \end{pmatrix}$

2) Take $v_2 = 0, v_3 = 1 \Rightarrow 4v_1 + 7 = 0 \Rightarrow v_1 = -\frac{7}{4} \Rightarrow$

another eigenvector can be taken as $v^2 = \begin{pmatrix} -\frac{7}{4} \\ 0 \\ 1 \end{pmatrix} \Rightarrow$

$e^{-4t}\begin{pmatrix} -\frac{1}{4} \\ 1 \\ 0 \end{pmatrix}$ & $e^{-4t}\begin{pmatrix} -\frac{7}{4} \\ 0 \\ 1 \end{pmatrix}$ are 2 independent

solutions

ii) Treating the simple (i.e. multiplicity 1) eigenvalue $\lambda = 8$: Find an eigenvector

$$(A-8I)v = \begin{pmatrix} -24 & -3 & -21 \\ 12 & -9 & 21 \\ 12 & 3 & 9 \end{pmatrix} v = 0$$

$$\left(\begin{array}{ccc|c} -24 & -3 & -21 & 0 \\ 12 & -9 & 21 & 0 \\ 12 & 3 & 9 & 0 \end{array}\right) \xrightarrow[\substack{R_2 \to 2R_2 + R_1 \\ R_1 \to -\frac{1}{3}R_1}]{R_3 \to R_3 - R_2} \left(\begin{array}{ccc|c} 8 & 1 & 7 & 0 \\ 0 & -21 & 21 & 0 \\ 0 & 12 & -12 & 0 \end{array}\right)$$

$$\xrightarrow[R_3 \to \frac{1}{12}R_3]{R_2 \to \frac{1}{21}R_2} \left(\begin{array}{ccc|c} 8 & 1 & 7 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right) \xrightarrow{R_3 \to R_3 + R_2} \left(\begin{array}{ccc|c} 8 & 1 & 7 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow 8v_1 + v_2 + 7v_3 = 0$
$\quad -v_2 + v_3 = 0$

Take $v_3 = 1 \Rightarrow v_2 = 1 \Rightarrow$
$8v_1 + 1 + 7 = 0 \Rightarrow v_1 = -1$

$\Rightarrow v^3 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector of $\lambda = 8 \Rightarrow e^{8t}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ is a solution

$\Rightarrow$ combining i) and ii) the general solution is

$$C_1 e^{-4t}\begin{pmatrix} -\frac{1}{4} \\ 1 \\ 0 \end{pmatrix} + C_2 e^{-4t}\begin{pmatrix} -\frac{7}{4} \\ 0 \\ 1 \end{pmatrix} + C_3 e^{8t}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

Problem 3

$$\begin{cases} x_1' = 11x_1 + 19x_2 - 14x_3 \\ x_2' = -15x_1 - 27x_2 + 22x_3 \\ x_3' = -12x_1 - 22x_3 + 19x_3 \end{cases}$$

(a) $-\lambda^3 + 3\lambda^2 - 4 = 0$ or $\lambda^3 - 3\lambda^2 + 4$

If λ is an integer root then it divides 4 $\Rightarrow$

λ can be only among $\pm 1, \pm 2, \pm 4$

Plug:

$\lambda = 1$: $-1 + 3 - 4 \neq 0$ ✗

$\boxed{\lambda = -1}$: $1 + 3 - 4 = 0$ ✓

$\boxed{\lambda = 2}$: $-8 + 12 - 4 = 0$ ✓

We found two roots $\lambda_1 = 1$ and $\lambda_2 = 2$

The third one can be found from the Viete formula for cubic equation

$$\underbrace{\lambda_1 \lambda_2}_{(-1)\,(2)} \lambda_3 = -\frac{\text{free term}}{\text{coeff of } \lambda^3} = -4$$

$\Rightarrow \lambda_3 = \frac{-4}{-2} = 2 \rightarrow 2$

Another way after finding $\lambda_1 = -1$, divide $-\lambda^3 + 3\lambda^2 - 4$ by $\lambda + 1$

$$\begin{array}{r|l} & -\lambda^2 + 4\lambda - 4 \\ \lambda + 1 & -\lambda^3 + 3\lambda^2 - 4 \\ & \underline{-\lambda^3 - \lambda^2} \\ & 4\lambda^2 \\ & \underline{4\lambda^2 + 4\lambda} \\ & -4\lambda - 4 \\ & \underline{-4\lambda - 4} \\ & 0 \end{array}$$

$\Rightarrow -\lambda^2 + 4\lambda - 4 = 0$

$\Leftrightarrow -(\lambda - 2)^2 = 0 \Rightarrow$

$\lambda_{2,3} = 2$

So, $\lambda=2$ is an eigenvalue of algebraic multiplicity 2
$\lambda=-1$ is an eigenvalue of algebraic multiplicity 1
($\Rightarrow$ its geometric multiplicity is 1)

It is remained to find the geometric multiplicity of $\lambda=2$.

$$(A-2I)v=\begin{pmatrix} 9 & 19 & -14 \\ -15 & -29 & 22 \\ -12 & -22 & 17 \end{pmatrix} v$$

$$\left(\begin{array}{ccc|c} 9 & 19 & -14 & 0 \\ -15 & -29 & 22 & 0 \\ -12 & -22 & 17 & 0 \end{array}\right) \underset{R_3\to 3R_3+4R_1}{\overset{R_2\to 3R_2+5R_1}{\sim}} \left(\begin{array}{ccc|c} 9 & 19 & -14 & 0 \\ 0 & -87+95 & 66-70 & 0 \\ 0 & -66+76 & 51-56 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 9 & 19 & -14 & 0 \\ 0 & 8 & -4 & 0 \\ 0 & 10 & -5 & 0 \end{array}\right) \underset{R_3\to\frac{1}{5}R_3}{\xrightarrow{R_2\to\frac{1}{4}R_2}} \left(\begin{array}{ccc|c} 9 & 19 & -14 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 2 & -1 & 0 \end{array}\right)$$

$$\xrightarrow{R_3\to R_3-R_2} \left(\begin{array}{ccc|c} 9 & 19 & -14 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \boxed{\text{geometric multiplicity is } 1}$$

The eigenspace is a line given by the system

$$\begin{cases} 9v_1+19v_2-4v_3=0 \\ 2v_2-v_3=0 \end{cases}$$

which geometrically is the intersection of two different planes.

(b) 1) Let us first treat $\lambda=2$. Since the geometric multiplicity of $\lambda=2$ is <u>strictly less</u> than the algebraic multiplicity we need to use the generalized eigenvectors (i.e. w)

We present two ways of solution according to algorithm 2 and algorithm 1 given in class

1.1 Using algorithm 2

i) Find an eigenvector v of $\lambda=2$

From the system at the end of page 9:

$$\begin{cases} 9v_1+19v_2-4v_3=0 \\ 2v_2-v_3=0 \end{cases}$$

Take $v_2=1 \Rightarrow 2-v_3=0 \Rightarrow v_3=2 \Rightarrow$

Plugging into the 1st eq. $\rightarrow$ $9v_1+19-28=0 \Rightarrow 9v_1-9=0 \Rightarrow v_1=1 \Rightarrow$

$$v=\begin{pmatrix}1\\1\\2\end{pmatrix} \text{ is an eigenvector}$$

ii) Find w such that

$(A-\lambda I)w=v$, where v is from the previous item

$$\begin{pmatrix}9 & 19 & -14\\ -15 & -29 & 22\\ -12 & -22 & 17\end{pmatrix} w = \begin{pmatrix}1\\1\\2\end{pmatrix}$$

Augmented matrix (doing the same sequence of elementary operators)

$$\left(\begin{array}{ccc|c}9 & 19 & -14 & 1\\ -15 & -29 & 22 & 1\\ -12 & -22 & 17 & 2\end{array}\right) \xrightarrow[R_3\to 3R_3+4R_1]{R_2\to 3R_2+5R_1} \left(\begin{array}{ccc|c}9 & 19 & -14 & 1\\ 0 & 8 & -4 & 8\\ 0 & 10 & -5 & 10\end{array}\right) \xrightarrow[R_3\to \frac{1}{5}R_3]{R_2\to \frac{1}{4}R_2}$$

$$\begin{pmatrix}9 & 19 & -14 & 1\\ 0 & 2 & -1 & 2\\ 0 & 2 & -1 & 2\end{pmatrix} \xrightarrow{R_3\to R_3-R_1} \left(\begin{array}{ccc|c}9 & 19 & -14 & 1\\ 0 & 2 & -1 & 2\\ 0 & 0 & 0 & 0\end{array}\right) \Rightarrow \begin{array}{l} 9w_1+19w_2-14w_3=1 \\ 2w_2-w_3=2 \end{array}$$

Take $w_3=0 \Rightarrow 2w_2-2=0 \Rightarrow w_2=1$

$\Rightarrow 9w_1+19=1 \Rightarrow w_1=-2$

$\Rightarrow w = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \Rightarrow$

$$e^{tA}v = e^{2t}\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \text{ and } e^{tA}w = e^{2t}(w+tv) = e^{2t}\left(\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}\right)$$

are two independent solutions

1.2) Using algorithm 1

From i) of page 10 the eigenline $E_2 = \left\{ c\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}; c \in \mathbb{R} \right\}$

Find the space of solutions of

$(A - \lambda I)^2 w = 0$ (i.e. the space $E_2^{(2)}$ in the notation of the class)

$$(A-2I)^2 = \begin{pmatrix} 9 & 19 & -14 \\ -15 & -29 & 22 \\ -12 & -22 & 17 \end{pmatrix}\begin{pmatrix} 9 & 19 & -14 \\ -15 & -29 & 22 \\ -12 & -22 & 17 \end{pmatrix} =$$

$$= \begin{pmatrix} 81-285+168 & 171-551+308 & -126+418-238 \\ -135+435-264 & -285+841-484 & 210-638+374 \\ -108+330-204 & -228+638-374 & 168-484+289 \end{pmatrix} = \underbrace{\begin{pmatrix} -36 & -72 & 54 \\ 36 & 72 & -54 \\ 18 & 36 & -27 \end{pmatrix}}_{\text{are multiple of each other}}$$

$(A-\lambda I)^2 w = 0 \rightarrow$ The augmented matrix is

$$\left(\begin{array}{ccc|c} -36 & -72 & 54 & 0 \\ 36 & 72 & -54 & 0 \\ 18 & 36 & -27 & 0 \end{array}\right) \underset{\substack{R_2 \to \frac{R_2}{18} \\ R_3 \to \frac{R_3}{9}}}{\overset{R_1 \to -\frac{R_1}{18}}{\sim}} \left(\begin{array}{ccc|c} 2 & 4 & -3 & 0 \\ 2 & 4 & -3 & 0 \\ 2 & 4 & -3 & 0 \end{array}\right) \underset{R_3 \to R_3 - R_1}{\overset{R_2 \to R_2 - R_1}{\sim}}$$

$$\sim \left(\begin{array}{ccc|c} 2 & 4 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Leftrightarrow 2w_1 + 4w_2 - 3w_3 = 0 \rightarrow \text{the equation of } E_2^{(2)}$$

choose $w_3 = 0$, $w_2 = 1 \Rightarrow 2w_1 + 4 = 0 \Rightarrow$
$\Rightarrow w_1 = -2$

So we make take $w = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$ (you can take any w in $E_2^{(2)}$ which is not in E_2)

ii) Calculate $v = (A - 1I)w = \begin{pmatrix} 9 & 19 & -4 \\ -15 & -29 & 22 \\ -12 & -22 & 17 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} =$

$$= \begin{pmatrix} -18+19 \\ 30-29 \\ 24-22 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

We got the same v and w as in the previous method

(we could get different v and w by another choice of w from $E_2^{(2)}$ in the previous step, but it will not change the general solution at the end)

2) Find an eigenvector for $\lambda = -1$

$$(A - (-1)I)v = \begin{pmatrix} 12 & 19 & -14 \\ -15 & -26 & 22 \\ -12 & -22 & 20 \end{pmatrix} v = 0$$

Augmented matrix

$$\left(\begin{array}{ccc|c} 12 & 19 & -14 & 0 \\ -15 & -26 & 22 & 0 \\ -12 & -22 & 20 & 0 \end{array}\right) \underset{R_3 \to R_3 + R_1}{\overset{R_2 \to 4R_2 + 5R_1}{\sim}} \left(\begin{array}{ccc|c} 12 & 19 & -14 & 0 \\ 0 & -104+95 & 88-70 & 0 \\ 0 & -3 & 6 & 0 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|c} 12 & 19 & -14 & 0 \\ 0 & -9 & 18 & 0 \\ 0 & -3 & 6 & 0 \end{array}\right) \underset{R_3 \to -\frac{1}{3}R_3}{\overset{R_2 \to -\frac{1}{9}R_2}{\sim}} \left(\begin{array}{ccc|c} 12 & 19 & -14 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right) \overset{R_3 \to R_3 - R_2}{\sim}$$

$$\sim \left(\begin{array}{ccc|c} 12 & 19 & -14 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{cases} 12v_1 + 19v_2 - 14v_3 = 0 \\ -v_2 + 2v_3 = 0 \end{cases}$$

Take $v_3 = 1 \Rightarrow v_2 = 2 \Rightarrow$

$12v_1 + 38 - 14 = 0 \Rightarrow 12v_1 = -24 \Rightarrow v_1 = -2$

$\Rightarrow \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$ is an eigenvector of $\lambda = -1 \Rightarrow e^{-t}\begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$ is a solution

 Combining all we get that the general solution

is

$$\boxed{X(t) = c_1 e^{2t}\begin{pmatrix}1\\1\\2\end{pmatrix} + c_2 e^{2t}\left(\begin{pmatrix}-2\\1\\0\end{pmatrix} + t\begin{pmatrix}1\\1\\2\end{pmatrix}\right) + c_3 e^{-t}\begin{pmatrix}-2\\2\\1\end{pmatrix}}$$

Problem 4 (Bonus)
$$x_1' = 6x_1 - 2x_2$$
$$x_2' = 5x_1 - x_2 - x_3$$
$$x_3' = -\tfrac{7}{2}x_1 + 3x_2 + 4x_3$$

$$A = \begin{pmatrix} 6 & -2 & 0 \\ 5 & -1 & -1 \\ -\frac{7}{2} & 3 & 4 \end{pmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} 6-\lambda & -2 & 0 \\ 5 & -1-\lambda & -1 \\ -\frac{7}{2} & 3 & 4-\lambda \end{vmatrix} = (6-\lambda)\left((\lambda-4)(\lambda+1) + 3\right) +$$

$$+2\left(5(4-\lambda) - \tfrac{7}{2}\right) = (6-\lambda)\left(\lambda^2 - 3\lambda - 1\right) + 2\left(\tfrac{33}{2} - 5\lambda\right) =$$

$$= \underline{6\lambda^2} - \underline{\underline{18\lambda}} - 6 - \lambda^3 + \underline{3\lambda^2} + \underline{\underline{\lambda}} + 33 - \underline{\underline{10\lambda}} = -\lambda^3 + 9\lambda^2 - 27\lambda + 27 =$$

$$= -(\lambda - 3)^3 = 0 \implies \lambda = 3 \text{ is the only eigenvalue}$$

and its algebraic multiplicity is $\boxed{3}$

For geometric multiplicity solve

$$(A - \lambda I)v = 0$$

$$(A - 3I)v = \begin{pmatrix} 3 & -2 & 0 \\ 5 & -4 & -1 \\ -\frac{7}{2} & 3 & 1 \end{pmatrix} v = 0$$

Augmented matrix

$$\left(\begin{array}{ccc|c} 3 & -2 & 0 & 0 \\ 5 & -4 & -1 & 0 \\ -\frac{7}{2} & 3 & 1 & 0 \end{array}\right) \xrightarrow[R_3 \to 3R_3 + \frac{7}{2}R_1]{R_2 \to 3R_2 - 5R_1} \left(\begin{array}{ccc|c} 3 & -2 & 0 & 0 \\ 0 & -2 & -3 & 0 \\ 0 & 2 & 3 & 0 \end{array}\right).$$

$$\underset{R_3 \to R_3 - R_2}{\sim} \left(\begin{array}{ccc|c} 3 & -2 & 0 & 0 \\ 0 & -2 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Leftrightarrow \begin{cases} 3v_1 - 2v_2 = 0 \\ 2v_3 + 3v_2 = 0 \end{cases} \quad (*)$$

geometric multiplicity is equal to 1.

(b) Using algorithm 1

We know that $E_\lambda \subset \underbrace{E_\lambda^{(2)}}_{\text{2-dimensional}} \subset E_\lambda^{(3)} = \mathbb{R}^3$ (here $\lambda = 3$)

Take any w which is not in the plane $E_\lambda^{(2)}$

We can find this plane by solving $(A - \lambda I)^2 v = 0$

Instead let us choose some "simple" w and check that it is indeed not in $E_\lambda^{(2)}$.

For example let $w = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

$$w^1 = (A - \lambda I)w = \begin{pmatrix} 3 & -2 & 0 \\ 5 & -4 & -1 \\ -\frac{7}{2} & 3 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

$$v = \underbrace{(A - \lambda I) w^1}_{= (A-\lambda I)^2 w} = \begin{pmatrix} 3 & -2 & 0 \\ 5 & -4 & -1 \\ -\frac{7}{2} & 3 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix} \neq 0 \quad \left(\begin{array}{l} \text{this indeed} \\ \text{shows that} \\ w \notin E_3^{(2)} \end{array}\right)$$

(by the way $\begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}$ is an eigenvector (plug it

in the system (*) on page 14.

So $\{e^{tA}v, e^{tA}w^1, e^{tA}w\}$ forms a fundamental

set of solutions

$$e^{tA}v = e^{\lambda t}v = e^{3t}\begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}$$

$(A-\lambda I)^2 w^1 = 0$ implies

$$e^{tA}w^1 = e^{\lambda t}(w^1 + t\underbrace{(A-\lambda I)w^1}_{v}) = e^{3t}\left(\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}\right)$$

$(A-\lambda I)^3 w = 0$ implies

$$e^{tA}w = e^{\lambda t}\left(w + t\underbrace{(A-\lambda I)w}_{w^1} + \frac{t^2}{2}\underbrace{(A-\lambda I)^2 w}_{v}\right) =$$

$$= e^{3t}\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + \frac{t^2}{2}\begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}\right)$$

$\Downarrow$

The general solution is

$$x(t) = c_1 e^{3t}\begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix} + c_2 e^{3t}\left(\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}\right) +$$

$$+ c_3 e^{3t}\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + \frac{t^2}{2}\begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}\right)$$

Rem You also can solve it using algorithms 2 or 3

For example, by algorithm 3, since in this case $(A-\lambda I)^3 = 0$

so $e^{At} = e^{\lambda t}\left(I + (A-\lambda I)t + \frac{1}{2}(A-\lambda I)^2 t^2\right)$ and you

plug in the data to get the fundamental matrix