Homework assignment 9, MATH 308, Solutions

Problem 11 (a)

(a) $2y''+7y'+6y=e^{5t}\sin 3t,\ y(0)=1,\ y'(0)=-1$

Apply Laplace transform to both sides

$$\mathcal{L}\{y\}=Y$$

$$\mathcal{L}\{y'\}=sY(s)-y(0)=sY(s)-1$$

$$\mathcal{L}\{y''\}=s^2Y(s)-sy(0)-y'(0)=s^2Y(s)-s+1$$

$$\mathcal{L}\{e^{5t}\sin 3t\}=\frac{3}{(s-5)^2+9}=\frac{3}{s^2-10s+34}$$

(here we use that $\mathcal{L}\{e^{\alpha t}\sin\beta t\}=\frac{\beta}{(s-\alpha)^2+\beta^2}$, $\alpha=5,\ \beta=3$)

$$\mathcal{L}\{2y''+7y'+6y\}=(2s^2+7s+6)Y(s)-2s+2-7=$$

$$=(2s^2+7s+6)Y(s)-2s-5=\frac{3}{(s-5)^2+9}$$

$$Y(s)=\boxed{\frac{1}{2s^2+7s+6}\left(\frac{3}{\underbrace{(s-5)^2+9}_{s^2-10s+34}}+2s+5\right)}=$$

$$=\frac{1}{2s^2+7s+6}\cdot\frac{2s^3-10s^2+68s+5s^2-50s+170+3}{s^2-10s+34}=\frac{2s^3-5s^2+18s+173}{(2s^2+7s+6)(s^2-10s+34)}$$

Problem 1 (B) $2y'' + 3y' - 5y = t^2 e^t$, $y(0) = 0$, $y'(0) = 1$

Apply Laplace transform to both sides

$\mathcal{L}\{y\} = Y(s)$

$\mathcal{L}\{y'\} = sY(s) - y(0) = sY(s)$

$\mathcal{L}\{y''\} = s^2 Y(s) - s\,y(0) - y'(0) = s^2 Y(s) - 1$

$\mathcal{L}\{t^2 e^t\} = \frac{2!}{(s-1)^3} = \frac{2}{(s-1)^3}$ (here we use that $\mathcal{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}}$, $a = 1$)

$\mathcal{L}\{2y'' + 3y' - 5y\} = (2s^2 + 3s - 5)Y(s) - 2 = \frac{2}{(s-1)^3} \Rightarrow$

$$Y(s) = \boxed{\frac{1}{2s^2+3s-5}\left(2 + \frac{2}{(s-1)^3}\right)} = \frac{2\left((s-1)^3+1\right)}{2s^2+3s-5}$$

Problem 2 (a) $F(s) = \dfrac{2s+5}{(s^2-2s-15)(s+3)}$

The denominator $Q(s) = \underbrace{(s^2-2s-15)}_{(s-5)(s+3)}(s+3) = (s-5)(s+3)^2$

$\Rightarrow$ the partial fraction decomposition is found in the form

$$\frac{2s+5}{(s-5)(s+3)^2} = \frac{A}{s-5} + \frac{B}{s+3} + \frac{C}{(s+3)^2} \Rightarrow$$

$$2s+5 = A(s+3)^2 + B(s-5)(s+3) + C(s-5)$$

To find C put $s=-3$: $-6+5 = C(-3-5) \Rightarrow$
$-8C = -1 \Rightarrow \boxed{C = \frac{1}{8}}$

To find A put $s=5$: $15 = A\cdot(5+3)^2 = A\,64 \Rightarrow$
$\boxed{A = \frac{15}{64}}$

To find B compare coefficient of s^2:
$0 = A+B \Rightarrow B = -A = \boxed{-\frac{15}{64}} \Rightarrow$

$$F(s) = \frac{15}{64(s-5)} - \frac{15}{64(s+3)} + \frac{1}{8(s+3)^2} \Rightarrow$$

$$\mathcal{L}^{-1}\{F(s)\} = \frac{15}{64}\mathcal{L}^{-1}\left\{\frac{1}{s-5}\right\} - \frac{15}{64}\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} + \frac{1}{8}\mathcal{L}^{-1}\left\{\frac{1}{(s+3)^2}\right\} =$$

$$= \boxed{\frac{15}{64}e^{5t} - \frac{15}{64}e^{-3t} + \frac{1}{8}te^{-3t}}$$

Problem 6

$$s^2-4s+53 = \underbrace{s^2-4s+4}_{(s-2)^2}+49 = (s-2)^2+49 = (s-2)^2+7^2$$

The partial fraction decomposition for $F(s)$ is found in the following form

$$\frac{3s+1}{(s^2-4s+53)(s-1)} = \frac{A(s-2)+7B}{(s-2)^2+7^2} + \frac{C}{s-1}$$

$$3s+1 = \left(A(s-2)+7B\right)(s-1) + C\left((s-2)^2+7^2\right)$$

To find C put $s=1$: $3+1 = C(1+49) = 50C$

$$\boxed{C = \frac{4}{50} = \frac{2}{25}}$$

To find B put $s=2$:

$$7 = \underbrace{7B\cdot(2-1)}_{7B} + 49C = 7B + \frac{98}{25} \Rightarrow$$

$$B = 1 - \frac{98}{25\cdot 7} = 1 - \frac{14}{25} = \boxed{\frac{11}{25}}$$

To find A compare coefficient of s^2:

$$0 = A + C \Rightarrow A = -C = \boxed{-\frac{2}{25}}$$

$$F(s) = -\frac{2}{25}\,\frac{s-2}{(s-2)^2+7^2} + \frac{11}{25}\,\frac{7}{(s-2)^2+7^2} + \frac{2}{25(s-1)}$$

$$\Downarrow$$

$$\mathcal{L}^{-1}\{F(s)\} = -\frac{2}{25}\,\mathcal{L}^{-1}\left\{\frac{s-2}{(s-2)^2+7^2}\right\} + \frac{11}{25}\,\mathcal{L}^{-1}\left\{\frac{7}{(s-2)^2+7^2}\right\} + \frac{2}{25}\,\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} =$$

$$= \boxed{-\frac{2}{25}\,e^{2t}\cos 7t + \frac{11}{25}\,e^{2t}\sin 7t + \frac{2}{25}\,e^{t}}$$

Problem 3

$$y'' + 2y' - 15y = 3e^{-2t}\cos 3t \quad , y(0) = 1, y'(0) = -1$$

$$\mathcal{L}\{y\} = Y$$

$$\mathcal{L}\{y'\} = sY(s) - y(0) = sY(s) - 1$$

$$\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - s + 1$$

$$\mathcal{L}\{y'' + 2y' - 15y\} = (s^2+2s-15)\,Y(s) - s + 1 - 2 =$$

$$= (s^2+2s-15)\,Y(s) - s - 1 = \mathcal{L}\{3e^{-2t}\cos 3t\} = 3\,\frac{s+2}{(s+2)^2+9}$$

$$\alpha = -2,\ \beta = 3$$

$$\Rightarrow \quad Y(s) = 3\,\frac{s+2}{((s+2)^2+9)\underbrace{(s^2+2s-15)}_{(s+5)(s-3)}} + \frac{s+1}{\underbrace{s^2+2s-15}_{(s+5)(s-3)}}$$

=

Decompose these two terms separately.

1. $$\frac{s+1}{(s+5)(s-3)} = \frac{A}{s+5} + \frac{B}{s-3}$$

$$s+1 = A(s-3) + B(s+5)$$

$s=3$: $4 = 8B \Rightarrow B = \frac{1}{2}$
$s=-5$: $-4 = -8A \Rightarrow A = \frac{1}{2}$ $\Rightarrow \frac{s+1}{(s+5)(s-3)} = \frac{1}{2}\frac{1}{s+5} + \frac{1}{2}\frac{1}{s-3}$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{s+1}{(s+5)(s-3)}\right\} = \frac{1}{2}e^{-5t} + \frac{1}{2}e^{3t} \qquad (*)$$

2. $$3\frac{s+2}{((s+2)^2+9)(s+5)(s-3)} = \frac{C(s+2)+3D}{(s+2)^2+9} + \frac{E}{s+5} + \frac{F}{s-3}$$

$$\Downarrow$$

$$3s+6 = \Big(C(s+2)+3D\Big)(s+5)(s-3) + E\big((s+2)^2+9\big)(s-3) +$$
$$+ F\big((s+2)^2+9\big)(s+5)$$

To find E put $s=-5$:

$$-9 = E\cdot(9+9)\,(-8) \Rightarrow 1 = E\cdot 16 \Rightarrow E = \frac{1}{16}$$

To find F put $s=3$

$$15 = F\,\underbrace{(25+9)}_{34}\,8 = 272F \Rightarrow F = \frac{15}{272}$$

To find D put $s=-2$:

$$0 = 3D\cdot 3\cdot(-5) + \frac{1}{16}\cdot 9\cdot(-5) + \frac{15}{\underbrace{272}_{17\cdot 16}}\cdot 9\cdot 3 =$$

$$= -45D - \frac{45}{16} + \frac{45}{16}\cdot\frac{9}{17} = -45\left(D + \frac{1}{16} - \frac{1}{16}\frac{9}{17}\right) =$$

$$= -45\left(D + \frac{1}{\cancel{16}_{2}} \cdot \frac{\cancel{8}}{17}\right) \Rightarrow \boxed{D = -\frac{1}{34}}$$

To find C compare coefficient of s^3:

$$0 = C + E + F = C + \frac{1}{16} + \frac{15}{16 \cdot 17} = C + \frac{1}{16}\left(1 + \frac{15}{17}\right) =$$

$$= C + \frac{1}{16} \cdot \frac{32}{17} = C + \frac{2}{17} \Rightarrow \boxed{C = -\frac{2}{17}}$$

$$\Downarrow$$

$$\mathcal{L}^{-1}\left\{3\frac{s+2}{\left((s+2)^2+9\right)(s+5)(s-3)}\right\} = -\frac{2}{17}e^{-2t}\cos 3t - \frac{1}{34}e^{-2t}\sin 3t +$$

$$+\frac{1}{16}e^{-5t} + \frac{15}{272}e^{3t} \quad (**)$$

Summing up $(*)$ and $(**)$ we get that

$$y(t) = -\frac{2}{17}e^{-2t}\cos 3t - \frac{1}{34}e^{-2t}\sin 3t + \left(\frac{1}{16} + \frac{1}{2}\right)e^{-5t} +$$

$$+\left(\frac{15}{272} + \frac{1}{2}\right)e^{3t} = \boxed{-\frac{2}{17}e^{-2t}\cos 3t - \frac{1}{34}e^{-2t}\sin 3t + \frac{9}{16}e^{-5t} + \frac{151}{272}e^{3t}}$$