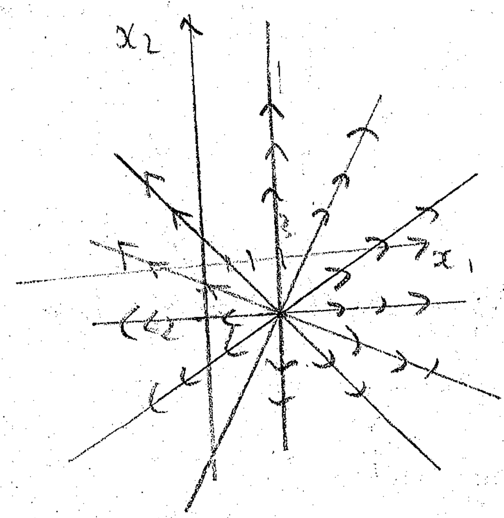
HW #9 Solutions MATH 308

$$\begin{cases} x_1' = 7x_1 - 21 \\ x_2' = 7x_2 + 14 \end{cases}$$

i) Critical points: $\begin{cases} 7x_1 - 21 = 0 \Rightarrow x_1 = 3 \\ 7x_2 + 14 = 0 \Rightarrow x_2 = -2 \end{cases} \Rightarrow \boxed{(3,-2) \text{ is the critical point}}$

Classification: $A = \begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix} = 7I \Rightarrow$ the eigenvalues are 7 with the algebraic and geometric multiplicity equal to 2 $\Rightarrow$ since it is positive $\Rightarrow$ $\boxed{\text{unstable}}$ $\boxed{\text{proper node}}$ (or star point)

ii) Phase portrait



Problem 2 $\begin{cases} x_1' = 3x_1 + 4x_2 \\ x_2' = -4x_1 - 3x_2 + 7 \end{cases}$

i) Critical points: $\begin{cases} 3x_1 + 4x_2 = 0 \\ -4x_1 - 3x_2 + 7 = 0 \end{cases} \Leftrightarrow \begin{cases} 3x_1 + 4x_2 = 0 & (Eq\ 1) \\ 4x_1 + 3x_2 = 7 & (Eq\ 2) \end{cases} \Rightarrow$

$4(Eq\ 2) - 3(Eq\ 1) \Rightarrow (16 - 9)x_1 = 28 \Rightarrow x_1 = 4 \overset{(Eq\ 1)}{\Rightarrow} 12 + 4x_2 = 0 \Rightarrow x_2 = -3 \Rightarrow$ (4,-3) the critical point

Classification $A = \begin{pmatrix} 3 & 4 \\ -4 & -3 \end{pmatrix}$, $tr A = 0$, $\det A = -9 + 16 = 7$. Char. equation is $\lambda^2 + 7 = 0 \Rightarrow \lambda = \pm i\sqrt{7}$ $\Rightarrow \boxed{\text{center}}$, $\boxed{\text{stable}}$

The eigenvalues are complex and on the imaginary axis

ii) Sketch of the phase portrait (without analysis of the shape of ellipses)

Since $a_{21} = -4 < 0$ the motion is clockwise (this is enough for this part)

Sketch for part ii)
(without analyzing shape of ellipses)

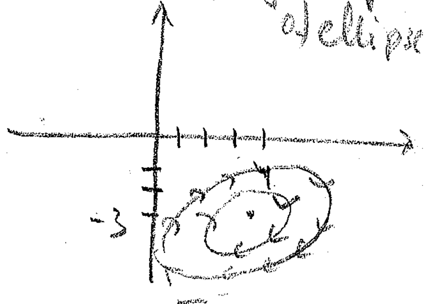


Fig 1

Sketch for part iii)
(after analyzing the shape of ellipses)

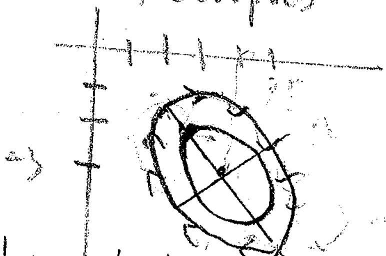


Fig 2

iii) (<u>bonus</u>) More accurate sketch, taking into account the shape of the ellipses

For this first find an eigenvector of $\lambda = \sqrt{7}i$

$$(A-\sqrt{7}i)v = \begin{pmatrix} 3-\sqrt{7}i & 4 \\ -4 & -3-\sqrt{7}i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow (3-\sqrt{7}i)v_1 + 4v_2 = 0$$

Take $v_1 = 4 \Rightarrow v_2 = -3+\sqrt{7}i \Rightarrow v = \begin{pmatrix} 4 \\ -3+\sqrt{7}i \end{pmatrix} = \underbrace{\begin{pmatrix} 4 \\ -3 \end{pmatrix}}_{a} + i\underbrace{\begin{pmatrix} 0 \\ \sqrt{7} \end{pmatrix}}_{b}$

(By the way, the shortest rotation from b to a is clockwise, confirming our conclusion in item ii)

The Gram matrix $\Gamma = \begin{pmatrix} a\cdot a & a\cdot b \\ a\cdot b & b\cdot b \end{pmatrix} = \begin{pmatrix} 4^2+3^2 & -3\sqrt{7} \\ -3\sqrt{7} & 7 \end{pmatrix} = \begin{pmatrix} 25 & -3\sqrt{7} \\ -3\sqrt{7} & 7 \end{pmatrix}$

Find the eigenvalues and the eigenvectors of Γ: $\operatorname{tr}\Gamma = 32$

$\det\Gamma = 175 - \overbrace{9\times 7}^{63} = 112$

The characteristic equation is $\lambda^2 - 32\lambda + 112 = 0 \Rightarrow D = 32^2 - 4\cdot 112 = 1024 - 448 = 576 \Rightarrow$

$\sqrt{D} = 24 \Rightarrow \lambda_1 = \frac{32-24}{2} = 4$, $\lambda_2 = \frac{32+24}{2} = 28 \Rightarrow \sqrt{\frac{\lambda_2}{\lambda_1}} = \sqrt{7} \approx 2.64$

(recall that by page 6 of notes of April 20 $\sqrt{\frac{\lambda_2}{\lambda_1}}$ represents the stretching scale of the ellipse along the eigenline of λ_2 (of Γ) compare to the eigenline of λ_1 and the eigenline of λ_1 (then the eigenline of λ_2 is orthogonal to it)

$(\Gamma - \lambda_1 I)v = \begin{pmatrix} 21 & -3\sqrt{7} \\ -3\sqrt{7} & 3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \Rightarrow -\sqrt{7}v_1 + v_2 = 0 \Rightarrow E_{\lambda_1} = c\begin{pmatrix} 1 \\ \sqrt{7} \end{pmatrix}$

$\Gamma \perp E_{\lambda_1}$ i.e. $E_{\lambda_2} = c\begin{pmatrix} -\sqrt{7} \\ 1 \end{pmatrix}$

The axis of symmetry are $\begin{pmatrix} 4 \\ -3 \end{pmatrix} + \sqrt{7}\begin{pmatrix} 0 \\ \sqrt{7} \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} = 4\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ & $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ for λ_1 & λ_2

See sketch 2 above

<u>oblem 3.</u> $\begin{cases} x_1' = 6x_1 - 2x_2 - 10 \\ x_2' = 9x_1 - 5x_2 - 7 \end{cases}$

i) Critical point: $\begin{cases} 6x_1 - 2x_2 - 10 = 0 \\ 9x_1 - 5x_2 - 7 = 0 \end{cases} \Leftrightarrow$

$\begin{cases} 3x_1 - x_2 = 5 & (Eq\,1) \\ 9x_1 - 5x_2 = 7 & (Eq\,2) \end{cases}$ $(Eq\,2) - 3(Eq\,1)$:

$\underbrace{-5x_2 + 3x_2}_{-2x_2} = 7 - 15 = -8 \Rightarrow x_2 = 4$

$\overset{Eq\,1}{\Rightarrow} 3x_1 - 4 = 5 \Rightarrow 3x_1 = 9 \Rightarrow x_1 = 3$

$\Rightarrow$ $\boxed{(3,4) \text{ is the critical point.}}$

Classifion

$A = \begin{pmatrix} 6 & -2 \\ 9 & -5 \end{pmatrix} \Rightarrow \operatorname{tr} A = 1, \ \det A = 6 \cdot (-5) + 18 = -12$

Characteristic equation: $\lambda^2 - \lambda - 12 = 0$ $\rightarrow$ <u>Rem</u>. Actually already from the fact that $\det A < 0$ we know that the critical point is saddle (see Rem on page 15 of notes of April 25)

$D = 1 + 48 = 49$

$\lambda_1 = \frac{1-7}{2} = -3$

$\lambda_2 = \frac{1+7}{2} = 4$

$\rightarrow$ eigenvalues are real and of opposite signs

$\Rightarrow$ $\boxed{(3,4) \text{ is a saddle point}}$ $\boxed{\text{unstable}}$

ii) <u>Sketch</u>. For more accurate sketch we need to find the eigenlines of A: a) For $\lambda_1 = -3$

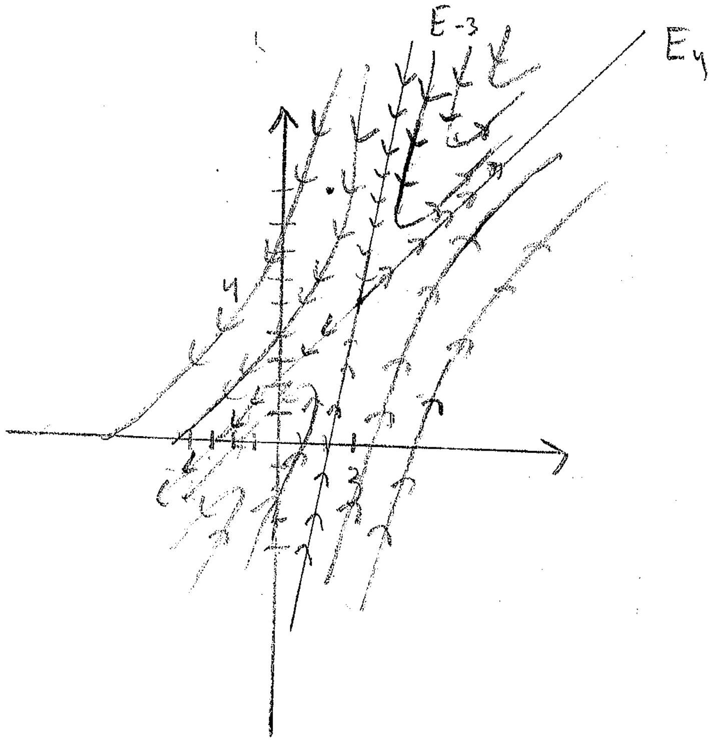
$(A + 3I)v = \begin{pmatrix} 9 & -2 \\ 9 & -2 \end{pmatrix} v = 0 \Rightarrow 9v_1 - 2v_2 = 0$

Take $v_1 = 2 \Rightarrow v_2 = 9 \Rightarrow$ the eigenline is generated by the vector $\begin{pmatrix} 2 \\ 9 \end{pmatrix}$

b) For $\lambda = 4$

$$(A - 4I)v = \begin{pmatrix} 2 & -2 \\ 9 & -9 \end{pmatrix} v = 0 \iff 2v_1 - 2v_2 = 0 \iff v_1 - v_2 = 0 \Rightarrow$$

If $v_2 = 1$, then $v_1 = 1 \Rightarrow$ the eigenline is generated by $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$



__oblem 4.__

$$\begin{cases} x_1' = 2x_1 + 2x_2 - 6 \\ x_2' = -3x_1 + 7x_2 + 39 \end{cases}$$

i) __Critical points:__ $\begin{cases} 2x_1 + 2x_2 - 6 = 0 \\ -3x_1 + 7x_2 + 39 = 0 \end{cases} \iff \begin{cases} x_1 + x_2 = 3 & (Eq\,1) \\ 3x_1 - 7x_2 = 39 & (Eq\,2) \end{cases}$

$(Eq\,2) - 3(Eq\,1)$: $-10x_2 = 30 \Rightarrow x_2 = -3 \xRightarrow{(Eq\,1)} x_1 - 3 = 3 \Rightarrow x_1 = 6 \Rightarrow$

$\boxed{(6,-3) \text{ is the critical point}}$

__Classification:__ $A = \begin{pmatrix} 2 & 2 \\ -3 & 7 \end{pmatrix}$, $\mathrm{tr}\,A = 9$, $\det A = 14 + 6 = 20$

Characteristic equation: $\lambda^2 - 9\lambda + 20 = 0$

$D = 81 - 80 = 1$

$\left.\begin{aligned} \lambda_1 &= \frac{9+1}{2} = 5 \\ \lambda_2 &= \frac{9-1}{2} = 4 \end{aligned}\right\}$ real positive $\Rightarrow$ $\boxed{\text{nodal source unstable}}$

ii) Sketch of the phase portrait

For more accurate sketch we need to find the eigenlines

a) The eigenline for $\lambda = 5$

$(A - 5I)v = \begin{pmatrix} -3 & 2 \\ -3 & 2 \end{pmatrix} v = 0 \Rightarrow -3v_1 + 2v_2 = 0 \Rightarrow$

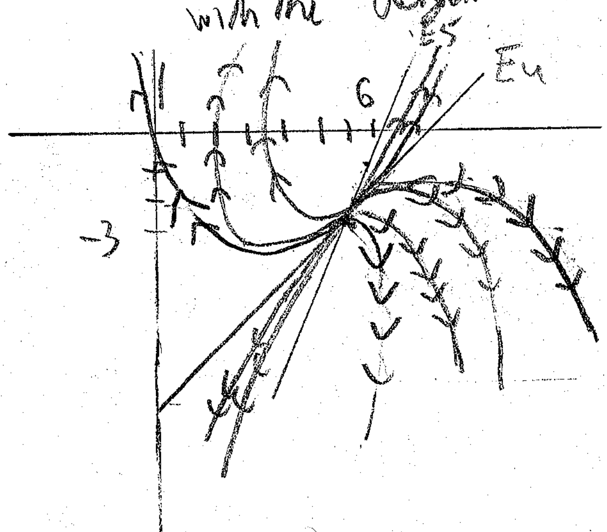
The eigenline E_5 is generated by $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

b) The eigenline for $\lambda = 4$

$(A - 4I)v = \begin{pmatrix} -2 & 2 \\ -3 & 3 \end{pmatrix} v = 0 \Rightarrow -v_1 + v_2 = 0 \Rightarrow$

The eigenline E_4 is generated by $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Since $5 > 4$ then all trajectories going out of the origin with the tangent line almost parallel to E_4



Problem 5 $\begin{cases} x_1' = -11x_1 - 2x_2 + 13 \\ x_2' = 2x_1 - 7x_2 + 5 \end{cases}$

i) Critical points : $\begin{cases} -11x_1 - 2x_2 + 13 = 0 \\ 2x_1 - 7x_2 + 5 = 0 \end{cases} \Leftrightarrow \begin{cases} 11x_1 + 2x_2 = 13 & (Eq\,1) \\ 2x_1 - 7x_2 = -5 & (Eq\,2) \end{cases}$

$2(Eq\,1) - 11(Eq\,2)$: $\underbrace{(4 + 77)}_{81} x_1 = 26 + 55 = 81 \Rightarrow x_1 = 1 \overset{(Eq\,1)}{\Rightarrow} 11 + 2x_2 = 13$

$\Rightarrow x_2 = 1 \Rightarrow$ $\boxed{(1,1) \text{ is the critical point}}$

Classification: $A = \begin{pmatrix} -11 & -2 \\ 2 & -7 \end{pmatrix} \Rightarrow \operatorname{tr} A = -18$, $\det A = 77 + 4 = 81 \Rightarrow$

Characteristic equation is $\lambda^2 + 18\lambda + 81 = 0 \Leftrightarrow (\lambda + 9)^2 = 0$
(or $D = 18^2 - 4 \cdot 81 = 324 - 324 = 0$)

$\Rightarrow \lambda_{1,2} = -9 \Rightarrow$ the eigenvalue $\lambda = -9$ of algebraic multiplicity 2, but of geometric multiplicity 1 (the latter because $A \neq -9I$) $\Rightarrow$ $\boxed{\text{improper node}}$, $\boxed{\text{asymptotically stable}}$

ii) Sketch of the phase portrait

a) Since $a_{21} = 2 > 0$ the motion for the part of the solution which is far of the origin will be <u>counterclockwise</u>
(for other way to determine this see * below)

b) Also for accuracy of the sketch, we need to find the eigenline of $\lambda = -9$

$$(A + 9I)v = \begin{pmatrix} -2 & -2 \\ 2 & 2 \end{pmatrix} v = 0 \Rightarrow v_1 + v_2 = 0 \Rightarrow \text{the eigenline is generated by } \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

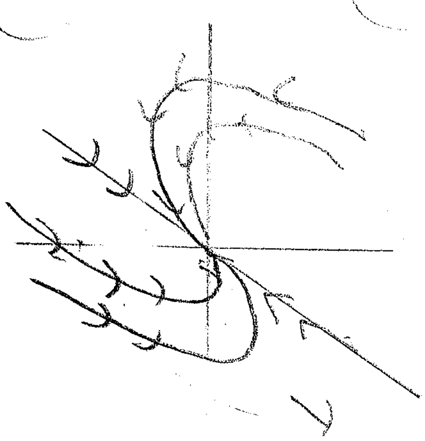


Fig 4

* Another way to determine the right shape of the trajectories is to find w s.t. $(A - \lambda I)w = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$-2w_1 - 2w_2 = 1 \Rightarrow w_2 = 0 \Rightarrow w_1 = -\frac{1}{2}$

$w = \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}$

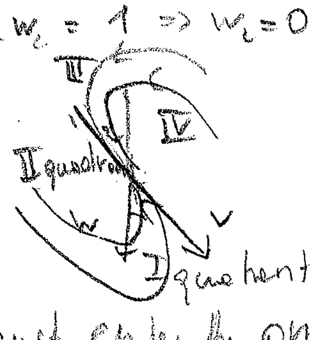


Fig 5

The trajectories must enter the origin from the 1st & 3rd quadrants as in the Fig 5 (that matches Fig 4)

Problem 6 $\begin{cases} x_1' = -8x_1 + 10x_2 \\ x_2' = -5x_1 + 2x_2 \end{cases}$

i) Critical points $\begin{cases} -8x_1 + 10x_2 = 0 \\ -5x_1 + 2x_2 = 0 \end{cases}$ $\det \begin{pmatrix} -8 & 10 \\ -5 & 2 \end{pmatrix} = -16+50 = 34 \neq 0$

$\Rightarrow x_1 = x_2 = 0$

$\boxed{\text{The only critical point is } (0,0)}$

Classification $A = \begin{pmatrix} -8 & 10 \\ -5 & 2 \end{pmatrix}$ $\operatorname{tr} A = -8+2 = -6$, $\det A = 34$ $\Rightarrow$

The characteristic equation is $\lambda^2 + 6\lambda + 34 = 0$

$D = 36 - 4 \cdot 34 = 36 - 136 = -100$

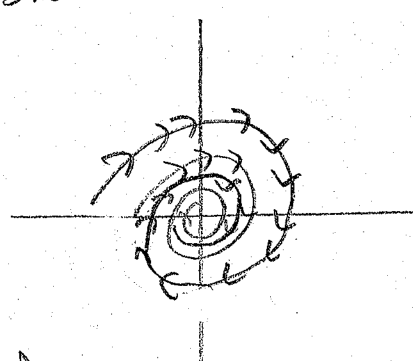
$\lambda_{1,2} = \frac{-6 \pm 10i}{2} = -3 \pm 5i \rightarrow$

complex, with negative real part $\Rightarrow$ $\boxed{\text{spiral sink}}$, $\boxed{\text{asymptotically stable}}$

ii) Sketch (without detailed shape)

$a_{21} = -5 < 0 \Rightarrow$ the rotation is clockwise

Sketch (sufficient for item ii)



iii) (bonus) More accurate sketch, taking into account the shape of the spirals

For this, first find an eigenvector of $\lambda = \lambda = -3+5i$

$$(A+(3-5i)I)v = \begin{pmatrix} -5-5i & 10 \\ -5 & 5-5i \end{pmatrix} v = 0 \iff$$

$$-(1+i)v_1 + 2v_2 = 0 \Rightarrow \text{ if } v_1 = 2 \text{ then } v_2 = 1+i \Rightarrow$$

$$v = \begin{pmatrix} 2 \\ 1+i \end{pmatrix} = \underbrace{\begin{pmatrix} 2 \\ 1 \end{pmatrix}}_{a} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{b} i$$

(by the way this gives another justification that the direction of rotation is clockwise, because the shortest direction of rotation from b to a is clockwise)

Consider the Gram matrix

$$\Gamma = \begin{pmatrix} a\cdot a & a\cdot b \\ a\cdot b & b\cdot b \end{pmatrix} = \begin{pmatrix} 2\cdot 2+1\cdot 1 & 1\cdot 1 \\ 1\cdot 1 & 1\cdot 1 \end{pmatrix} = \begin{pmatrix} 5 & 1 \\ 1 & 1 \end{pmatrix}$$

Find the eigenvalues of Γ: $\operatorname{tr}\Gamma = 6$, $\det\Gamma = 5-1=4$

Char. equation of Γ is $\lambda^2 - 6\lambda + 4 = 0$

$D = 36 - 4\cdot 4 = 20$, $\sqrt{D} = 2\sqrt{5}$

$\lambda_1 = \frac{6 - 2\sqrt{5}}{2} = 3-\sqrt{5}$

$\lambda_2 = 3+\sqrt{5}$

$$\Rightarrow \frac{\lambda_2}{\lambda_1} = \frac{3+\sqrt{5}}{3-\sqrt{5}} = \frac{(3+\sqrt{5})^2}{(3-\sqrt{5})(3+\sqrt{5})} = \frac{9+5+6\sqrt{5}}{9-5} = \frac{7+3\sqrt{5}}{2}$$

$\sqrt{\frac{\lambda_2}{\lambda_1}} \approx 2.62 \rightarrow$ the coefficient of the stretch of the spirals along E_{λ_2} compared to E_{λ_1}

Find an eigenline of E_{λ_1} for Γ (then $E_{\lambda_2} \perp E_{\lambda_1}$)

$$(\Gamma - \lambda_1 I)v = \begin{pmatrix} 5-3+\sqrt{5} & 1 \\ 1 & 1-3+\sqrt{5} \end{pmatrix} v = \begin{pmatrix} 2+\sqrt{5} & 1 \\ 1 & -2+\sqrt{5} \end{pmatrix} v$$

$\Rightarrow (2+\sqrt{5})v_1 + v_2 = 0 \Rightarrow E_{\lambda_1}$ is generated by $\begin{pmatrix} -1 \\ 2+\sqrt{5} \end{pmatrix} \Rightarrow$

E_{λ_2} is generated by $\begin{pmatrix} 2+\sqrt{5} \\ 1 \end{pmatrix}$

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Then the axes of symmetries of ellipses (before multiplication by exponentially decaying factors)

$$\tilde{a} = -a + (2+\sqrt{5})b = \begin{pmatrix} -2 \\ -1+2+\sqrt{5} \end{pmatrix} = \begin{pmatrix} -2 \\ 1+\sqrt{5} \end{pmatrix}$$

slope is $-\frac{1+\sqrt{5}}{2} \approx -1.62$

$$\tilde{b} = (2+\sqrt{5})a + b = \begin{pmatrix} 4+2\sqrt{5} \\ 3+\sqrt{5} \end{pmatrix}$$ which is parallel to $\begin{pmatrix} 1+\sqrt{5} \\ 2 \end{pmatrix}$ (and perpendicular to $\begin{pmatrix} -2 \\ 1+\sqrt{5} \end{pmatrix}$)

slope is $\frac{2}{1+\sqrt{5}} \approx 0.62$

Sketch