Homework #9 Solutions, MATH308 - SUMMER 2012

Problem 1 According to the method of reduction of order we look for the second solution in the form

$$y(t) = v(t)y_1(t) = v(t)t^{-3}$$

Substitute into our equation

$$\begin{array}{r|l} 12\times & y(t) = v(t)\,t^{-3} \\ + \; 8t\times & y'(t) = -3v(t)\,t^{-4} + v'(t)t^{-3} \\ + \; t^2\times & y''(t) = 12v(t)t^{-5} - 6v'(t)t^{-4} + v''(t)t^{-3} \end{array}$$

$$t^2y'' + 8ty' + 12y = \underbrace{(12-24+12)}_{0}t^{-3}v + \underbrace{(-6+8)}_{2}t^{-2}v' + v''t^{-1} = 0$$

$$t^{-1}v'' + 2t^{-2}v' = 0 \Leftrightarrow$$

$$v'' + 2t^{-1}v' = 0$$

Substitution: $w = v'$

$$w' + 2t^{-1}w = 0 \Leftrightarrow w' = -2t^{-1}w \Leftrightarrow w = \tilde{C}_1 e^{-\int 2t^{-1}dt} =$$

$$= \tilde{C}_1 e^{-2\ln t} = \tilde{C}_1 t^{-2} \Rightarrow v = \int c_1 t^{-2}dt = \underbrace{-\tilde{C}_1}_{c_1} t^{-1} + c_2 \Rightarrow$$

$$y(t) = v(t)t^{-3} = c_1t^{-4} + c_2t^{-3}$$

So, we can take $\boxed{y_2(t) = t^{-4}}$ as a second solution

Problem 2 (a) i) $\alpha = -4$; characteristic equation is

$$r^2 + 5r + 6 = 0 \Leftrightarrow (r+2)(r+3) = 0 \Rightarrow r_1 = -2,\ r_2 = -3$$

So $\alpha = -4$ is not a root of the char. equation $\Rightarrow$ the multiplicity $s = 0 \Rightarrow$ we look for a particular solution in the form

$$y(t) = Ae^{-4t}$$

ii) $\alpha = -4$, $\beta = 3$, the char. polynomial is as in the previous items $\Rightarrow$ $-4+3i$ is not its root $\Rightarrow$ the multiplicity is equal to $s=0$ $\Rightarrow$ we look for a particular solution in the form

$$y(t) = e^{-4t}(A_1 \cos 3t + A_2 \sin 3t).$$

iii) $\alpha = -2$; the char. polynomial is as in the item i) $\Rightarrow$ α is a root of char. polynomial but not repeated root $\Rightarrow$ the multiplicity s of α is equal to 1, $s=1$ $\Rightarrow$ we look for a particular solution in the form

$$y(t) = Ate^{-2t}.$$

iv) $\alpha + i\beta = -2+10i$; the characteristic equation is as in the item i) $\Rightarrow$ $\alpha+i\beta$ is <u>not</u> a root of char. polynomial $\Rightarrow s=0 \Rightarrow$ we look for a particular solution in the form

$$y(t) = e^{-2t}(A_1 \cos 10t + A_2 \sin 10t).$$

(v) $\alpha = 7$; the char. equation is $r^2 - 12r + 36 = 0 \Leftrightarrow (r-6)^2 = 0 \Rightarrow r_{1,2} = 6$ $\Rightarrow \alpha = 7$ is not a root of char. polynomial $\Rightarrow s=1 \Rightarrow$ We look for a particular solution in the form $y(t) = Ae^{7t}$

(vi) A particular solution can be found as the sum of particular solutions of two equations

$$y'' - 12y' + 36y = 5e^{6t} \quad (1) \text{ and}$$

$$y'' - 12y' + 36y = -2e^{5t}\sin 2t \quad (2)$$

For the equation (1) $\alpha = 6$ and the characteristic polynomial is as in the item (v) $\Rightarrow$ $\alpha = 6$ is <u>a root and α is a repeated root</u> $\Rightarrow$ $s=2$ $\Rightarrow$ for (1) we look for a particular solution in the form $y_1(t) = At^2 e^{6t}$

For the equation (2)

$\alpha + i\beta = 5+2i \Rightarrow \alpha + i\beta$ is not a root of char. polynomial $\Rightarrow$

for (2) we look for a particular solution in the form

$$y_2(t) = e^{5t}(A_1 \cos 2t + A_2 \sin 2t)$$

Therefore for the original equation we look for a particular solution in the form

$$At^2 e^{6t} + e^{5t}(A_1 \cos 2t + A_2 \sin 2t)$$

(VII) If $\omega^2 \neq \omega_0^2$, then $\alpha = i\omega$ is not a root of char. polynomial $\Rightarrow$

$s=0 \Rightarrow Y_p(t) = A_1 \cos \omega t + A_2 \sin \omega t$,

If $\omega^2 = \omega_0^2$, then $\alpha = i\omega$ is a root of char. polynomial but not a repeated root $\Rightarrow s = 1 \Rightarrow$

$$Y_p(t) = t(A_1 \cos \omega t + A_2 \sin \omega t)$$

(VIII) $\alpha + i\beta = 3+4i$, the characteristic polynomial is

$$r^2 - 6r + 25 = 0$$

$$D = 36 - 100 = -64 \Rightarrow$$

$$r_{1,2} = \frac{6 \pm 8i}{2} = 3 \pm 4i \Rightarrow$$

$3+4i$ is a root of char. polynomial but not a repeated root $\Rightarrow s=1 \Rightarrow$ We look for a particular solution in the form

$$y(t) = e^{3t} t (A_1 \cos 4t + A_2 \sin 4t)$$

Problem 2 b) As mentioned in item a ii) we look for a solution in the form

$$y(t) = e^{-4t}(A_1 \cos 3t + A_2 \sin 3t)$$

Substidute into equation

$$6\times \quad y(t) = A_1 e^{-4t}\cos 3t + e^{-4t} A_2 \sin 3t$$

$$5\times \quad y'(t) = -4A_1 e^{-4t}\cos 3t + 3A_2 e^{-4t}\cos 3t - 4A_2 e^{-4t}\sin 3t - 3A_1 e^{-4t}\sin 3t =$$

$$= (-4A_1 + 3A_2)e^{-4t}\cos 3t + (-4A_2 - 3A_1)e^{-4t}\sin 3t$$

$$1\times \quad y''(t) = \left(-4(-4A_1+3A_2) + 3(-4A_2-3A_1)\right)\cos 3t + (-4)(-4A_2-3A_1)e^{-4t}\sin 3t +$$

$$+ (-3)(-4A_1+3A_2)e^{-4t}\sin 3t =$$

$$= (16A_1 - 12A_2 - 12A_2 - 9A_1)e^{-4t}\cos 3t + (16A_2 + 12A_1 + 12A_1 - 9A_2)e^{-4t}\sin 3t$$

$$= (7A_1 - 24A_2)e^{-4t}\cos 3t + (24A_1 + 7A_2)e^{-4t}\sin 3t$$

$$y'' + 5y' + 6y = (7A_1 - 24A_2 + 5(-4A_1 + 3A_2) + 6A_1)e^{-4t}\cos 3t +$$

$$+ (24A_1 + 7A_2 - 5(4A_2 + 3A_1) + 6A_2)e^{-4t}\sin 3t$$

$$= (7A_1 - 24A_2 - 20A_1 + 15A_2 + 6A_1)e^{-4t}\cos 3t + (24A_1 + 7A_2 - 20A_2 - 15A_1 +$$

$$+ 6A_2)e^{-4t}\sin 3t = (-7A_1 - 9A_2)e^{-4t}\cos 3t +$$

$$(9A_1 - 7A_2)e^{-4t}\cos 3t = 3e^{-4t}\sin 3t \Rightarrow \text{(comparing}$$

coefficient)

$$\begin{cases} -7A_1 - 9A_2 = 0 & \times 9 \\ 9A_1 - 7A_2 = 3 & \times 7 \end{cases} \Rightarrow 7\times Eq\,2 + 9\,Eq\,1 =$$

$$= (63-63)A_1 - (49+81)A_2 = 21 \Rightarrow -130A_2 = 21 \Rightarrow A_2 = -\frac{21}{130}$$

$$\Rightarrow A_1 = -\frac{9}{7}A_2 = -\frac{9}{7}\left(-\frac{21}{130}\right) = \frac{27}{130} \Rightarrow$$

$$Y_p(t) = e^{-4t}\left(\frac{27}{130}\cos 3t - \frac{21}{130}\sin 3t\right) \Rightarrow$$

The gen. solution

$$\boxed{y(t) = e^{-4t}\left(\frac{27}{130}\cos 3t - \frac{21}{130}\sin 3t\right) + C_1 e^{-2t} + C_2 e^{-3t}}$$

Problem 2 c) As follows from item 1(e) v11

Case 1 If $\omega_0^2 \neq \omega^2$ we look for a particular solution in the form

$$y(t) = A_1 \cos\omega t + A_2 \sin\omega t \Rightarrow$$

$$y'(t) = \omega A_2 \cos\omega t - A_1 \omega \sin\omega t$$

$$y''(t) = -A_1\omega^2 \cos\omega t - A_2 \omega^2 \sin\omega t$$

$$\Rightarrow y'' + \omega_0^2 y = \left(-A_1\omega^2 + A_1\omega_0^2\right)\cos\omega t + \left(-A_2\omega^2 + A_2\omega_0^2\right)\sin\omega t =$$

$$= A_1\left(\omega_0^2 - \omega^2\right)\cos\omega t + A_2\left(\omega_0^2 - \omega^2\right)\sin\omega t = \cos\omega t \Rightarrow$$

$$A_1\left(\omega_0^2 - \omega^2\right) = 1 \Rightarrow A_1 = \frac{1}{\omega_0^2 - \omega^2}$$

$$A_2\left(\omega_0^2 - \omega^2\right) = 0 \Rightarrow A_2 = 0 \qquad \Rightarrow Y_p(t) = \frac{\cos\omega t}{\omega_0^2 - \omega^2} \Rightarrow$$

gen solution is $y(t) = \dfrac{\cos\omega t}{\omega_0^2 - \omega^2} + C_1 \cos\omega_0 t + C_2 \sin\omega_0 t$

Case 2 If $\omega_0^2 = \omega^2$ then we look for a particular solution in the form

$$y(t) = A_1 t\cos\omega t + A_2 t\sin\omega t$$

$$y'(t) = A_2 \omega t\cos\omega t - A_1\omega t\sin\omega t + A_1\cos\omega t + A_2 \sin\omega t$$

$$y''(t) = -A_1\omega^2 t\cos\omega t - A_2\omega^2 t\sin\omega t - 2A_1\omega\sin\omega t + 2A_2\omega\cos\omega t$$

$$\Rightarrow y'' + \omega^2 y(t) = \underbrace{\left(\omega_0^2 - \omega^2\right)}_{=0}\left(A_1 t\cos\omega t + A_2 t\sin\omega t\right) - 2A_1\omega\sin\omega t + 2A_2\omega\cos\omega t =$$

$$= -2A_1\omega\sin\omega t + 2A_2\omega\cos\omega t = \cos\omega t \Rightarrow \begin{cases} -2A_1 = 0 \Rightarrow A_1 = 0 \\ 2A_2\omega = 1 \Rightarrow A_2 = \dfrac{1}{2\omega} \end{cases} \Rightarrow$$

$Y_p(t) = \dfrac{t}{2\omega}\sin\omega t \Rightarrow$ the general solution

$$\boxed{y(t) = \frac{t}{2\omega}\sin\omega t + C_1\cos\omega t + C_2\sin\omega t} \quad (\text{here } \omega = \pm\omega_0)$$