10.2: SERIES

a_k is called general(common) term of the series

A series is a sum of sequence:

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \ldots + a_n + \ldots$$

For a given sequence[1] $\{a_k\}_{k=1}^{\infty}$ define the following: $\{S_n\}_{n=1}^{\infty}$

$S_1 = a_1$

$S_2 = S_1 + a_2 = a_1 + a_2$

$S_3 = S_2 + a_3 = a_1 + a_2 + a_3$

$S_4 = S_3 + a_4 = a_1 + a_2 + a_3 + a_4$

$$\boxed{S_n = S_{n-1} + a_n = \sum_{k=1}^{n} a_k}$$

[1] $k = 1$ for convenience, it can be anything

The s_n's are called partial sums and they form a sequence $\{s_n\}_{n=1}^{\infty}$.
We want to consider the limit of $\{s_n\}_{n=1}^{\infty}$:

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k \sim \sum_{k=1}^{\infty} a_k$$

If $\{s_n\}_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series $\sum_{k=1}^{n} a_k$ is *convergent*. The number s is called the **sum** of the series.[2]

If $\{s_n\}_{n=1}^{\infty}$ is divergent then the series $\sum_{k=1}^{\infty} a_k$ is *divergent*.

[2]When we write $\sum_{k=1}^{\infty} a_k = s$ we mean that by adding sufficiently many terms of the series we can get as close as we like to the number s.

GEOMETRIC SERIES

$$a + ar + ar^2 + \ldots + ar^{n-1} + \ldots = \sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^{n-0} \qquad (a \neq 0)$$

Each term is obtained from the preceding one by multiplying it by the *common ratio* r.

$-1 < r < 1$

FACT: The geometric series is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}.$$

$r \geq 1$, $r \leq -1$

If $|r| \geq 1$, the geometric series is divergent.

EXAMPLE 1. *Determine whether the following series converges or diverges. If it is converges, find the sum. If it is diverges, explain why.*

(a) $\sum_{n=1}^{\infty} 5 \cdot \left(\frac{2}{7}\right)^n = \sum_{n=1}^{\infty} 5 \cdot \left(\frac{2}{7}\right)^{n-1} \cdot \frac{2}{7} = \sum_{n=1}^{\infty} \frac{10}{7} \cdot \left(\frac{2}{7}\right)^{n-1}$

geometric $r = \frac{2}{7}$, $a = \frac{10}{7}$

$\boxed{\text{convergent}}$ b/c $|r| < 1$

$S = \frac{a}{1-r} = \frac{10/7}{1-2/7} = \frac{10}{5} = \boxed{2}$

(b) $\sum_{n=0}^{\infty} \frac{(-4)^{3n}}{5^{n-1}} = \sum_{n=0}^{\infty} \frac{(-4^3)^n}{5^n \cdot 5^{-1}} = \sum_{n=0}^{\infty} 5 \cdot \left(-\frac{64}{5}\right)^n$

geometric series with $r = -\frac{64}{5} < -1$ } divergent

(c) $1 - \frac{3}{2} + \frac{9}{4} - \frac{27}{8} + \dots = \sum_{n=0}^{\infty} \left(-\frac{3}{2}\right)^n$

$a = 1$, $r = -\frac{3}{2} < -1 \Rightarrow$ divergent

geometric

(d) $\sum_{n=1}^{\infty} 4^{n+1} \cdot 9^{2-n} = \sum_{n=1}^{\infty} 4^{n+1} \cdot 9^{-(n-2)} = \sum_{n=1}^{\infty} \frac{4^{n+1}}{9^{n-2}}$

$2-n = -(n-2)$

$9^{2-n} = 9^{-(n-2)}$

$$= \sum_{n=1}^{\infty} \frac{4^{n-1} \cdot 4^2}{9^{n-1} \cdot 9^{-1}}$$

$$= \sum_{n=1}^{\infty} 16 \cdot 9 \left(\frac{4}{9}\right)^{n-1}$$

$$= \sum_{n=1}^{\infty} 144 \left(\frac{4}{9}\right)^{n-1}$$

Geometric $\left.\begin{array}{l} a = 144 \\ r = \frac{4}{9} \end{array}\right\}$ $\boxed{\text{convergent}}$ b/c $|r|<1$

$$S = \frac{a}{1-r} = \frac{144}{1-4/9} = \boxed{\frac{1296}{5}}$$

EXAMPLE 2. *Write the number* $.\overline{17}$ *as a ratio of integers.*

$$.\overline{17} = \frac{m}{n}$$

$$.\overline{17} = .1717171717171717\ldots = .17 + .0017 + .000017 + .00000017 + \ldots = .17 + .17 \cdot 10^{-2} + .17 \cdot 10^{-4} + .17 \cdot 10^{-6} + \ldots$$

Geometric series with $a = .17$

$r = 10^{-2} = 0.01$

convergent b/c $|r| < 1$

$$.\overline{17} = \frac{a}{1-r} = \frac{.17}{1-0.01} = \boxed{\frac{17}{99}}$$

TELESCOPING SUM

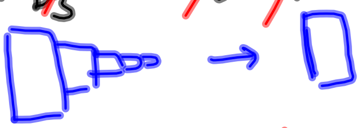
Let b_n be a given sequence. Consider the following series:

$$\sum_{n=1}^{\infty} \underbrace{(b_n - b_{n+1})}_{a_n = b_n - b_{n+1}}$$

Partial sum

$$S_n = a_1 + a_2 + \dots + a_n$$

$$= b_1 - \cancel{b_2} + \cancel{b_2} - \cancel{b_3} + \cancel{b_3} - \cancel{b_4} + \dots + \cancel{b_{n-1}} - \cancel{b_n} + \cancel{b_n} - b_{n+1}$$



$$= b_1 - b_{n+1}$$

The sum of telescoping series (if it converges)

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty}(b_1 - b_{n+1})$$

$$S = b_1 - \lim_{n\to\infty} b_{n+1}$$

Question: Are these series telescoping?

$$\sum_{n=1}^{\infty} b_{n+1} - b_n \qquad \sum b_n - b_{n-1}$$

$$\sum b_{n+2} - b_n \qquad \sum b_n - b_{n+3}$$

EXAMPLE 3. *Determine whether the following series converges or diverges. If it is converges, find the sum. If it is diverges, explain why.*

(a) $\displaystyle\sum_{n=1}^{\infty}\left(\sin\frac{1}{n}-\sin\frac{1}{n+1}\right) = \sum_{n=1}^{\infty} b_n - b_{n+1}$ Telescoping

$b_n = \sin\frac{1}{n}$

$S_n = b_1 - \lim_{n\to\infty} b_{n+1} = \sin 1 - \lim_{n\to\infty} \underbrace{\sin\frac{1}{n+1}}_{\to 0} = \sin 1$

$\boxed{\text{convergent and } S = \sin 1}$

(b) $\displaystyle\sum_{n=1}^{\infty}\ln\frac{n+1}{n+2} = \sum_{n=1}^{\infty} \underbrace{\ln(n+1)}_{b_n} - \underbrace{\ln(n+2)}_{b_{n+1}}$ Telescoping divergent

$S = b_1 - \lim_{n\to\infty} b_{n+1} = \ln 2 - \lim_{n\to\infty} \ln(n+2) = -\infty$

(c) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$

Use part. fraction decomp.

$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} = \underbrace{\frac{1}{n}}_{= b_n} - \underbrace{\frac{1}{n+1}}_{= b_{n+1}}$ Telescoping

$1 = A(n+1) + Bn$

$n = -1 \Rightarrow B = -1$

$n = 0 \Rightarrow A = 1$

$S = \lim_{n\to\infty} S_n = b_1 - \lim_{n\to\infty} b_{n+1} = 1 - \lim_{n\to\infty} \frac{1}{n+1} = \boxed{1}$ sum

convergent

THEOREM 4. *If the series* $\sum_{n=1}^{\infty} a_n$ *is convergent, then* $\lim_{n\to\infty} a_n = 0$.

REMARK 5. The converse is not necessarily true.

THE TEST FOR DIVERGENCE:

If $\lim_{n\to\infty} a_n$ *does not exist or if* $\lim_{n\to\infty} a_n \neq 0$, *then the series* $\sum_{n=1}^{\infty} a_n$ *is divergent.*

REMARK 6. If you find that $\lim_{n\to\infty} a_n = 0$ then the Divergence Test fails and thus another test must be applied.

geometric

series

telescoping

otherwise

Divergence Test

$\lim_{n\to\infty} a_n \neq 0 \Rightarrow \sum a_n$ diverg

$\lim_{n\to\infty} a_n = 0 \Rightarrow$ use tests from the sections we are going to learn

EXAMPLE 7. *Use the test for Divergence to determine whether the series diverges.*

(a) $\sum_{n=1}^{\infty} \underbrace{\frac{n^2}{3(n+1)(n+2)}}_{a_n}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{3(n+1)(n+2)} = \frac{1}{3} \neq 0$$

The series diverges

(b) $\sum_{n=1}^{\infty} \cos\frac{\pi n}{2}$

$\lim_{n\to\infty} \cos\frac{\pi n}{2}$ DNE because

n is odd $\Rightarrow \cos\frac{\pi n}{2} = 0$

n is even $\Rightarrow \cos\frac{\pi n}{2} = \pm 1$

(oscilating)

The series diverges

(c) $\sum_{n=1}^{\infty} \underbrace{\frac{(-1)^n}{n^2}}_{a_n}$

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{1}{n^2} = 0$$

$$\Downarrow$$

$$\lim_{n\to\infty} a_n = 0$$

Divergence Test Fails here. Thus, to make a conclusion we have to use some other test. (See Next Sections)