10.3 deals with positive series only

partial sums $\{S_n\}$ positive & increasing

bounded from below

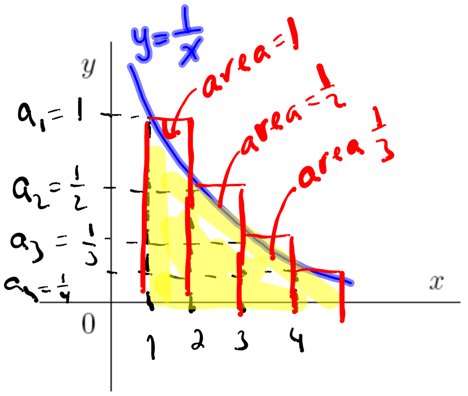
10.3: The Integral and Comparison Tests; Estimating Sums

QUESTION: For what values of p the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent?

If $p = 1$ then $\sum_{n=1}^{\infty} \frac{1}{n}$ is called harmonic series.

$f(x) = \frac{1}{x}$, $x \geq 1$

$f(n) = \frac{1}{n}$



$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n} + \dots$$

$$\int_1^{\infty} f(x)\,dx = \underbrace{\int_1^{\infty} \frac{dx}{x}}_{\text{divergent}} < 1 + \frac{1}{2} + \frac{1}{3} + \dots = \sum_{n=1}^{\infty} \frac{1}{n}$$

$\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent

THE INTEGRAL TEST *Suppose f is a continuous, positive, decreasing function on* $[1, \infty)$ *and let* $a_n = f(n)$. *Then the series* $\sum_{n=1}^{\infty} a_n$ *is convergent if and only if the improper integral* $\int_1^{\infty} f(x)\,dx$ *is convergent.* In other words:

If $\int_1^{\infty} f(x)\,dx$ converges $\Rightarrow$ $\sum_{n=1}^{\infty} a_n$ converges

If $\int_1^{\infty} f(x)\,dx$ diverges $\Rightarrow$ $\sum_{n=1}^{\infty} a_n$ diverges

Note that one can consider interval $[a, \infty)$ instead of $[1, \infty)$ if $\sum_{n=a}^{\infty} a_n$.

FACT: The p-series, $\sum_{n=1}^{\infty} \frac{1}{n^p}$, converges if $p > 1$ and diverges if $p \leq 1$.

EXAMPLE 1. *Determine if the following series is convergent or divergent:*

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1000}{n\sqrt{n}}$

$f(x) = \dfrac{1000}{x\sqrt{x}} = \dfrac{1000}{x^{3/2}}$, $x \geq 1$

On $[1, \infty)$ $f(x)$ is positive ✓

continuos ✓

decreasing ✓

$f'(x) = \left(\dfrac{1000}{x^{3/2}}\right)' = 1000\left(-\dfrac{3}{2}\right)x^{-\frac{5}{2}} < 0$

We can apply Int. Test

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{1000}{x^{3/2}}\,dx$$

Anti-derivat

$$1000\,\frac{x^{-\frac{3}{2}+1}}{-\frac{3}{2}+1} = -\frac{2000}{\sqrt{x}}$$

$$= \lim_{t\to\infty} \int_1^t \frac{1000\,dx}{x^{3/2}} = -2000 \lim_{t\to\infty} \left.\frac{1}{\sqrt{x}}\right|_1^t = -2000 \lim_{t\to\infty} \frac{1}{\sqrt{t}} - 1$$

$= 2000 \Rightarrow$ the integral converges

$\Downarrow$

the series converge

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

$f(x) = \dfrac{1}{x \ln x} \quad x \geq 2$

On $[2, \infty)$

$f(x)$ is positive ✓
continuous ✓
decreasing ✓

$\left(\begin{array}{l} x \text{ is increasing} \\ \ln x \text{ is increasing} \end{array} \right\} \Rightarrow \dfrac{1}{x} \cdot \dfrac{1}{\ln x}$ is decr.

Apply Int. Test

$$\int_2^{\infty} f(x)\,dx = \int_2^{\infty} \frac{dx}{x \ln x} = \lim_{t \to \infty} \int_2^{t} \frac{dx}{x \ln x}$$

$$\int \frac{dx}{x \ln x} = \int \frac{du}{u} = \ln|u| = \ln(\ln x)$$

$u = \ln x \Rightarrow du = \dfrac{dx}{x}$

$$= \lim_{t \to \infty} \ln(\ln(x)) \Big|_2^t$$

$$= \lim_{t \to \infty} \underbrace{\ln(\ln t)}_{\infty} - \ln(\ln 2))$$

The integral diverges $\Rightarrow$ the given series diverges

A disadvantage of the Integral Test: it does force us to do improper integrals which are in some cases may be impossible to determine the convergence of. For example, consider

$$\sum_{n=0}^{\infty} \frac{1}{\underbrace{4^n + n^4}}$$

positive

Try to apply Int. Test

$$f(x) = \frac{1}{4^x + x^4}$$

what is anti-derivative of $f(x) = ?$

Partial sum

$$S_n = \sum_{k=0}^{n} \frac{1}{4^k + k^4}$$

$\{S_n\}$ is positive (= bounded from below by 0)

$\{S_n\}$ increasing sequence, because $S_{n+1} = S_n + \overset{>0}{a_{n+1}} > S_n$

$$S_n = \sum_{k=0}^{n} \frac{1}{4^k + k^4} < \sum_{k=0}^{n} \frac{1}{4^k} < \sum_{k=0}^{\infty} \frac{1}{4^k} = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$$

geometric

$r = \frac{1}{4}$

$a = 1$

$S = \frac{a}{1-r}$

$S_n \le \frac{4}{3} \Rightarrow S_n$ is bdd above

Conclusion $\{S_n\}$ is monotonic and bounded $\Rightarrow$

$\Rightarrow \{S_n\}$ is convergent

$\Rightarrow$ the given series $\boxed{\text{converges}}$

THE COMPARISON TEST *Suppose that* $\sum a_n$ *and* $\sum b_n$ *are series with positive terms and* $a_n \le b_n$ *for all* n.

- *If* $\sum b_n$ *is convergent then* $\sum a_n$ *is also convergent.*
- *If* $\sum a_n$ *is divergent then* $\sum b_n$ *is also divergent.*

EXAMPLE 2. *Determine if the following series is convergent or divergent:*

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^4+4}{n^6+6}$ $\quad \sim \sum \frac{n^4}{n^6} = \sum \frac{1}{n^2}$ conv.

$$\frac{n^4+4}{n^6+6} \le \frac{n^4+4}{n^6} = \frac{1}{n^2} + \frac{4}{n^6}$$

$$\left.\begin{array}{l} \sum \frac{1}{n^2} \\ \sum \frac{4}{n^6} \end{array}\right\} \text{conv. as p-series } p>1 \quad \Rightarrow \sum \frac{1}{n^2} + \frac{4}{n^6} \text{ converges}$$

$\Downarrow$ by Comp. Test

the given series converges

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{n^{2000}}{n^{2001} - \sin^{2000} n}$ >0 $\sim \sum \frac{n^{2000}}{n^{2001}} = \sum \frac{1}{n}$

divergent

$$\frac{n^{2000}}{n^{2001} - \sin^{2000} n} \geq \frac{n^{2000}}{n^{2001}} = \frac{1}{n}$$

The series $\sum \frac{1}{n}$ is divergent $(p=1)$

$\Rightarrow$ By Comp. Test the given series $\boxed{\text{is divergent}}$

(c) $\sum_{n=1}^{\infty} \frac{\cos^4 n}{n^2\sqrt{n}}$

$\sim \sum \frac{1}{n^2\sqrt{n}} = \underbrace{\sum \frac{1}{n^{5/2}} \text{ converges } p = 5/2}$

$$\frac{\cos^4 n}{n^2\sqrt{n}} \le \frac{1}{n^2\sqrt{n}} = \frac{1}{n^{5/2}}$$

By Comp. Test the given series converges

(d) $\sum_{n=1}^{\infty} \frac{5^n + 1}{4^n}$ $= \sum_{n=1}^{\infty} \frac{5^n}{4^n} + \frac{1}{4^n} = \sum \left(\frac{5}{4}\right)^n + \frac{1}{4^n}$

$\sum \left(\frac{5}{4}\right)^n$ $\sum \frac{1}{4^n}$

geometric series

$r = \frac{5}{4} > 1$ $r = \frac{1}{4}$

div. conv.

$\frac{5^n+1}{4^n} = \left(\frac{5}{4}\right)^n + \frac{1}{4^n} > \left(\frac{5}{4}\right)^n$

$\sum_{n=1}^{\infty} \left(\frac{5}{4}\right)^n$ diverges

$\Bigg\}\Rightarrow$ By Comp. Test the given series diverges

(e) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{4^n + n^4}$

$$\left.\begin{array}{l} \dfrac{1}{4^n+n^4} \leq \dfrac{1}{4^n} \\ \displaystyle\sum_{n=0}^{\infty} \frac{1}{4^n} \text{ converges} \end{array}\right\} \Rightarrow \text{by Comp. Theorem } \sum_{n=0}^{\infty} \frac{1}{4^n+n^4} \text{ converges}$$

2nd way

$$\frac{1}{4^n+n^4} < \frac{1}{n^4}$$

$\sum \frac{1}{n^4}$ conv. p-series $p=4>1$

WARNING: Distinguish between $\underbrace{\displaystyle\sum_{n=1}^{\infty} n^p}_{\text{p-series}}$ and $\underbrace{\displaystyle\sum_{n=1}^{\infty} p^n}_{\text{geom. series with } r=p}$.

$$\sum \frac{1}{n^{-p}}$$

In some cases inequalities are useless. For example, for the series

$$\sum_{n=0}^{\infty} \frac{1}{4^n - n}$$

we have

$$\frac{1}{4^n - n} > \frac{1}{4^n}$$

But $\sum \frac{1}{4^n}$ convergent

geometric

$-1 < r = \frac{1}{4} < 1$

$\Rightarrow$ no conclusion can be made using Comparison Test.

$\Downarrow$

use another test

THE LIMIT COMPARISON TEST *Suppose that* $\sum a_n$ *and* $\sum b_n$ *are series with positive terms. If*

$$\lim_{n\to\infty} a_n \cdot \frac{1}{b_n} = \lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c *is a finite number and* $c > 0$*, then either both series converge or both diverge.*

$c=0$ or $c=\infty \Rightarrow$ TEST FAILS $\Rightarrow$ use other test

EXAMPLE 3. *Determine if the following series is convergent or divergent:*

(a) $\sum_{n=1}^{\infty} \underbrace{\frac{1}{4^n - n}}_{a_n} \sim \sum \underbrace{\frac{1}{4^n}}_{b_n}$ convergent as geom. series $r=\frac{1}{4}<1$

$$c = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{4^n}{4^n - n} \overset{\frac{\infty}{\infty}}{=} \lim_{n\to\infty} \frac{4^n \ln 4 / 4^n}{(4^n \ln 4 - 1)/4^n}$$

L'Hôpital's Rule

$$= \lim_{n\to\infty} \frac{\ln 4}{\ln 4 - \frac{1}{4^n}} = \frac{\ln 4}{\ln 4 - 0} = 1 > 0$$

The given series converges by LCT.

(b) $\sum_{n=2}^{\infty} \underbrace{\frac{n^2 + n}{\sqrt{n^5 + n^3}}}_{a_n} \sim \sum \frac{n^2}{\underbrace{\sqrt{n^5}}_{b_n}} = \sum \frac{n^2}{n^{5/2}} = \sum \frac{1}{n^{1/2}}$ divergent p-series with $p=\frac{1}{2}<1$

$$c = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} a_n \cdot \frac{1}{b_n} = \lim_{n\to\infty} \frac{(n^2+n)\sqrt{n}}{\sqrt{n^5+n^3}} =$$

$$= \lim_{n\to\infty} \sqrt{\frac{(n^2+n)^2\, n}{n^5 + n^3}} = 1 > 0$$

By LCT the given series $\boxed{\text{diverges}}$

Illustration: Why c in the Limit Comparison Test must be positive and finite:

$c > 0 \Rightarrow$ LCT

$$\left.\begin{array}{l} c = 0 \\ c = \infty \\ c \text{ DNE (see Ex. 5A)} \end{array}\right\} \text{LCT fails}$$

$c < 0$ (impossible b/c we apply LCT only to positive series)

For example, $\sum \underbrace{\frac{1}{n}}_{a_n}$ div. and $\sum \underbrace{\frac{1}{n^2}}_{b_n}$ conv.

$$c = \lim_{n\to\infty} a_n \cdot \frac{1}{b_n} = \lim_{n\to\infty} \frac{n^2}{n} = \lim_{n\to\infty} n = \infty \Rightarrow \text{LCT fails}$$

For example, $\sum \frac{1}{n^2}$ conv. $\sum \frac{1}{n^3}$ conv.

$c = \infty$

$\sum \frac{1}{n^4}$ conv. $\sum \frac{1}{n^2}$ conv. $\Rightarrow c = 0$

If $\sum_{n=1}^{\infty} a_n$ converges $\Rightarrow$

$$S = \sum_{n=1}^{\infty} a_n = \underbrace{a_1 + a_2 + a_3 + \dots + a_{n-1} + a_n}_{\text{partial sum } \; S_n} + \underbrace{a_{n+1} + a_{n+2} + a_{n+3} + \dots}_{R_n \quad \text{remainder}}$$

$$\boxed{S = S_n + R_n}$$

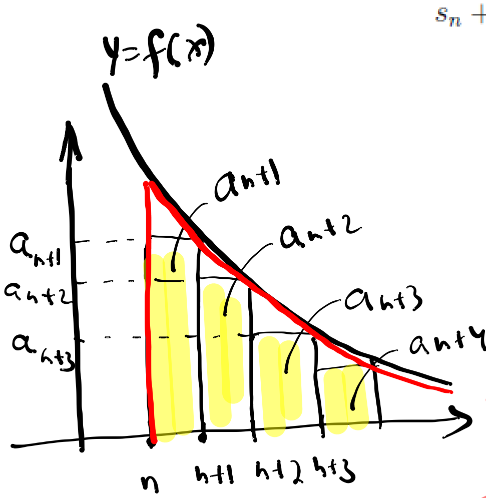
REMAINDER ESTIMATE FOR THE INTEGRAL TEST

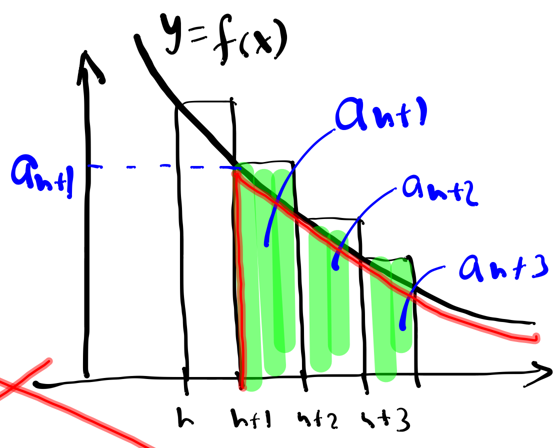
If $\sum a_n$ converges by the Integral Test and $R_n = s - s_n$, then

$$\boxed{\int_{n+1}^{\infty} f(x)\,\mathrm{d}x \leq \underset{S-S_n}{\underset{\parallel}{R_n}} \leq \int_{n}^{\infty} f(x)\,\mathrm{d}x}$$

which implies

$$s_n + \int_{n+1}^{\infty} f(x)\,\mathrm{d}x \leq s \leq s_n + \int_{n}^{\infty} f(x)\,\mathrm{d}x$$





$$\int_{n+1}^{\infty} f(x)\,dx \le R_n = a_{n+1} + a_{n+2} + a_{n+3} + \dots \le \int_{n}^{\infty} f(x)\,dx$$

EXAMPLE 4. *Given* $\sum_{n=1}^{\infty} \frac{1}{n^3}$ convergent p-series, $p = 3 > 1$

(a) *Approximate the sum of the series by using the sum of the first* 10 *terms* (Use Calculator)

$$S_{10} = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \dots + \frac{1}{10^3} \approx 1.1975$$

(b) *Estimate the error* $= R_{10}$

$$\int_{11}^{\infty} f(x)\,dx \le R_{10} \le \int_{10}^{\infty} f(x)\,dx, \text{ where } f(x) = \frac{1}{x^3}$$

$$\int_{n}^{\infty} \frac{dx}{x^3} = \lim_{t\to\infty} \int_{n}^{t} \frac{dx}{x^3} = \lim_{t\to\infty} -\frac{1}{2x^2}\Big|_{n}^{t} = -\frac{1}{2}\lim_{t\to\infty}\left(\underbrace{\frac{1}{t^2}}_{0} - \frac{1}{n^2}\right) = \boxed{\frac{1}{2n^2}}$$

$$\int_{11}^{\infty} \frac{dx}{x^3} \le R_{10} \le \int_{10}^{\infty} \frac{dx}{x^3}$$

$$\frac{1}{2\cdot 11^2} \le R_{10} \le \frac{1}{2\cdot 10^2}$$

$$\boxed{\frac{1}{242} \le R_{10} \le \frac{1}{200}}$$ Final answer

$$\Downarrow$$

$$0.00413 \lesssim R_{10} \le 0.005$$

$$\Downarrow$$

$$1.1975 + 0.00413 \lesssim S = S_{10} + R_{10} \le 1.1975 + 0.005$$

$$\Downarrow$$

$$1.2016 \lesssim S \le 1.2025$$ sum estimate with the error R_{10}.

(c) *How many terms are required to ensure that the sum is accurate to within* 0.0005*?*

Find n such that

$$R_n \le \int_{n}^{\infty} \frac{dx}{x^3} = \frac{1}{2n^2} \le 0.0005$$

$$\frac{1}{2n^2} \le 5\cdot 10^{-4}$$

$$1 \le 5\cdot 10^{-4} n^2 \cdot 2$$

$$1 \le 10\cdot 10^{-4} n^2$$

$$1 \le 10^{-3} n^2$$

$$10^3 \le n^2$$

$$n \ge \sqrt{1000} \approx 31.6$$

Conclusion,

We need $\boxed{32}$ terms to ensure that the sum is accurate to within 0.0005.

EXAMPLE 5. *Given*

$$\underbrace{\sum_{n=1}^{\infty} \frac{1+\sin n}{2n^3}}_{a_n}. \sim \sum \frac{2}{2n^3} = \underbrace{\sum \frac{1}{n^3}}_{b_n} \text{ Conv.}$$

(a) *Prove the convergence.*

$$c = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1+\sin n}{2\cancel{n^3}} \cdot \cancel{n^3} = \lim_{n\to\infty} \frac{\overset{\text{oscillating}}{1+\sin n}}{2} = \text{DNE}$$

$\Rightarrow$ LCT fails $\Rightarrow$ use another test

Use Comparison Test $\quad 0 \le \dfrac{1+\sin n}{2n^3} \le \dfrac{2}{2n^3} = \dfrac{1}{n^3}$

$\sum \frac{1}{n^3}$ converges $\overset{CT}{\Rightarrow}$ the given series converges.

(b) *By comparison the series to a p-series, estimate the error in using s_{100} to approximate the sum of the series.*

By Example 4 for the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ we have

$$R_n \le \int_n^{\infty} \frac{dx}{x^3} = \frac{1}{2n^2}$$

$$\Downarrow$$

$$R_{100} \le \frac{1}{2\cdot 100^2} = 0.5\cdot 10^{-4}$$

Thus the error r_{100} for the given series $\sum_{n=1}^{\infty} \frac{1+\sin n}{n^3}$

will be $r_{100} \le R_{100} \le \boxed{0.5\cdot 10^{-4}}$