10.4: Other Convergence Tests

Alternating Series Test

DEFINITION 1. *An alternating series is a series whose terms are alternately positive and negative. It means if* $\sum a_n$ *is an alternating series then either*

$$a_n = (-1)^n b_n, \quad b_n > 0$$

$b_n = |a_n| > 0$

or

$$a_n = (-1)^{n+1} b_n, \quad b_n > 0.$$

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \underbrace{(-1)^n b_n}_{\text{general term}} = -b_1 + b_2 - b_3 + b_4 - \dots.$$

OR

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{n+1} b_n = b_1 - b_2 + b_3 - b_4 - \dots.$$

Alternating Series Test: *If* $\lim_{n\to\infty} b_n = 0$ *and the sequence* $\{b_n\}$ *is decreasing then the series* $\sum(-1)^n b_n$ *is convergent.*

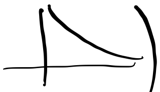
REMARK 2. This test will only tell us when a series converges and not if a series will diverge.

EXAMPLE 3. *Determine if the following series are convergent or divergent:*

(a) $\sum_{n=1}^{\infty} \underbrace{\frac{(-1)^{n+1}}{n}}_{a_n} = \sum_{n=1}^{\infty} (-1)^{n+1} \underbrace{\frac{1}{n}}_{b_n}$ Alternating Series

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$

$\{b_n\}$ is decreasing ($f(x)=\frac{1}{x}$



)

$\Big\}$ AST $\Longrightarrow$ The series converges

(b) $\sum_{n=1}^{\infty} \frac{\cos \pi n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ Alternating Series

$\cos \pi n = \begin{cases} 1, & n \text{ is even} \\ -1, & n \text{ is odd} \end{cases}$

$b_n = \frac{1}{\sqrt{n}}$ decreasing

$b_n \xrightarrow[n\to\infty]{} 0$

By AST the series converges.

Note here that $\sum \frac{1}{n}$ and $\sum \frac{1}{\sqrt{n}}$ are divergent $(p \le 1)$

(c) $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^2+5}$ Alternating

$b_n = \frac{n^2}{n^2+5}$

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n^2}{n^2+5} = 1 \neq 0$

AST fails

Try Divergence Test

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^n n^2}{n^2+5} = \begin{cases} \lim_{n\to\infty} \frac{n^2}{n^2+5} = 1 & \text{if } n \text{ is even} \\ \lim_{n\to\infty} -\frac{n^2}{n^2+5} = -1 & \text{if } n \text{ is odd} \end{cases}$$

$\Rightarrow \lim_{n\to\infty} a_n$ DNE $\Rightarrow$ the series diverges by Divergent Test

Absolute Convergence

- A series $\sum a_n$ is called absolutely convergent if the series of absolute values $\sum |a_n|$ is convergent.

- If a series $\sum a_n$ is convergent but the series of absolute values $\sum |a_n|$ is divergent then the series $\sum a_n$ is **conditionally convergent**.

For example:

- The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges absolutely, because $\sum \left| \frac{(-1)^n}{n^2} \right| = \sum \frac{1}{n^2}$ converges

- The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges conditionally, because $\sum \left| \frac{(-1)^n}{n} \right| = \sum \frac{1}{n}$ diverges

FACT: If $\sum a_n$ *converges absolutely then it is also convergent.*

EXAMPLE 4. *Determine if each of the following series are absolutely convergent, conditionally convergent or divergent.*

(a) $\sum_{n=1}^{\infty} \frac{1}{n^4}$ positive series ⇒ it coincides with the series of abs. values: $\sum \left|\frac{1}{n^4}\right| = \sum \frac{1}{n^4}$ → abs. conv.

convergent ⇒ abs. conv.

Note: Any convergent positive series is also absolutely convergent.

(a) $\sum_{n=1}^{\infty} \frac{\sin n}{n^3}$

not positive series
not alternating series } ⇒ use abs. conv.

$\boxed{\sum_{n=1}^{\infty} \left|\frac{\sin n}{n^3}\right| \text{ converges}}$ by Comparison Test:

$\left|\frac{\sin n}{n^3}\right| \le \frac{1}{n^3}$ and $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges

⇓

$\sum_{n=1}^{\infty} \frac{\sin n}{n^3}$ conv. absolutely (in particular, converge

Conclusion $\boxed{\text{absolutely converges}} \Rightarrow \boxed{\text{converges}}$

For positive series: $\boxed{\text{abs. converges}} = \boxed{\text{converges}}$

Remainder Estimate

The Alternating Series Theorem. *If $\sum_{n=1}^{\infty}(-1)^n b_n$ is a convergent alternating series and you used a partial sum s_n to approximate the sum s (i.e. $s \approx s_n$) then*

$$\boxed{|R_n| \le b_{n+1}.} \Leftrightarrow -b_{n+1} \le R_n \le b_{n+1}$$

$$\sum_{n=1}^{\infty}(-1)^n b_n = \underbrace{\overbrace{\overbrace{\underbrace{-b_1}_{s_1} + b_2}^{s_2} - b_3}^{s_3} + b_4 - b_5 + \dots + (-1)^n b_n}_{s_n} \underbrace{+ \dots}_{R_n}$$

where $b_n > 0$

↓

converges $\Rightarrow$ $\lim_{n\to\infty} s_n = s$ (series sum)

⇓

$\{b_n\}$ decreasing

$|R_n| = |s - s_n|$

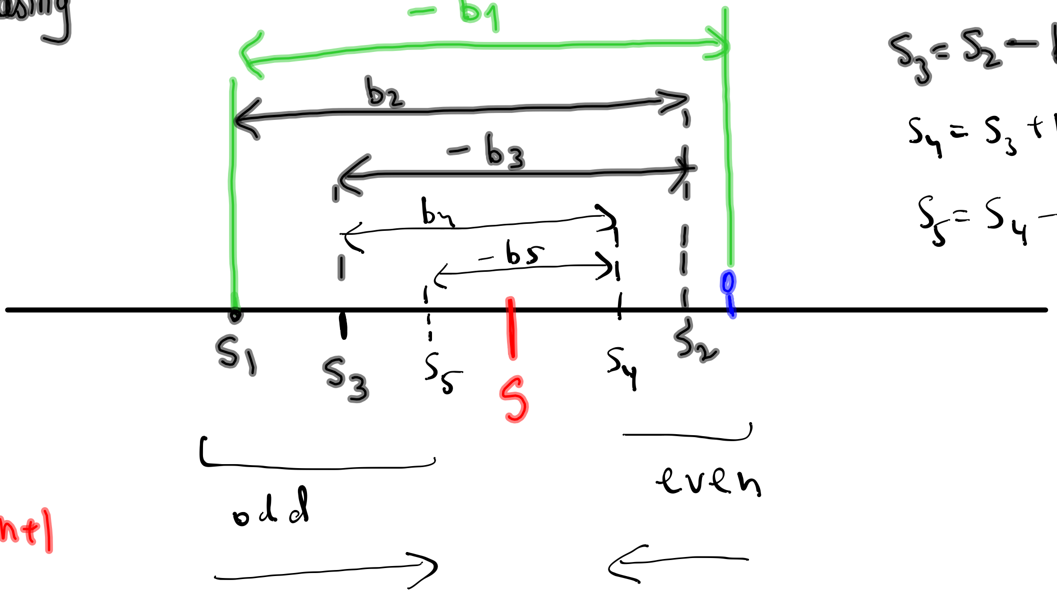
$s_1 = -b_1$

$s_2 = s_1 + b_2$

$s_3 = s_2 - b_3$

$s_4 = s_3 + b_4$

$s_5 = s_4 - b_5$



$R_n = s - s_n$

$|R_n| \le b_{n+1}$

$|R_1| \le b_1$

$|R_n| = |s - s_n| \le b_{n+1}$

EXAMPLE 5. *Given* $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$. $= \sum_{n=1}^{\infty} \underbrace{(-1)^n \frac{1}{n^3}}_{a_n}$, $b_n = \frac{1}{n^3}$

(a) *Show that the series converges. Does it converge absolutely?*

Alternating Series

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n^3} = 0$

$\left\{\frac{1}{n^3}\right\} \downarrow$ $\Downarrow$ By AST the series converges,

$\sum \frac{1}{n^3}$ convergent (p-series $p = 3 > 1$)

$\Downarrow$

The series converges absolutely

$\Downarrow$

converges

2nd proof for convergence.

(b) *Use* s_6 *to approximate the sum of the series.*

$S_6 = a_1 + a_2 + \dots + a_6 = -1 + \frac{1}{2^3} - \frac{1}{3^3} + \frac{1}{4^3} - \frac{1}{5^3} + \frac{1}{6^3} \approx -0.89978$

(c) *Determine the upper bound on the error in using* s_6 *to approximate the sum.*

$|R_n| \le b_{n+1}$

$|R_6| \le b_{6+1} = b_7 = \boxed{\frac{1}{7^3} \approx 0.0029}$

Note (b) & (c) $\Rightarrow$ sum $S = S_6 + R_6 \approx -0.89978 \pm 0.0029$

EXAMPLE 6. *Given* $\sum_{n=1}^{\infty} \frac{(-1)^{n+3}}{n^5}$.

$b_n = \frac{1}{n^5}$

(a) *Show that the series converges.*

$\sum \frac{1}{n^5}$ converges as p-series $(p = 5 > 1)$

$\Rightarrow$ the given series converges absolutely $\Rightarrow$ converges

OR use AST

strictly less

(b) *Approximate the sum of the series with error less than* 10^{-5}.

Find n such that $|R_n| < 10^{-5}$

We know $|R_n| \leq b_{n+1}$

$$\underbrace{|R_n| \leq \frac{1}{(n+1)^5} < 10^{-5}}$$

solve this inequality

$$\frac{1}{(n+1)^5} < \frac{1}{10^5}$$

$$(n+1)^5 > 10^5$$

$$n+1 > 10 \Rightarrow \boxed{n > 9}$$

$$S \approx S_{10} = \frac{1}{1} - \frac{1}{2^5} + \frac{1}{3^5} - \frac{1}{4^5} + \frac{1}{5^5} + \dots - \frac{1}{10^5} \approx 0.972116$$

Note $S = S_{10} + R_{10} \approx 0.972116 \pm 10^{-5}$

arbitrary series

RATIO TEST For a series $\sum a_n$ define

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} |a_{n+1}| \cdot \frac{1}{|a_n|}$$

- If $L < 1$ then the series is absolutely convergent (which implies the series is convergent.)
- If $L > 1$ then the series is divergent. (Note $L = \infty > 1$)
- If $L = 1$ then the series may be divergent, conditionally convergent or absolutely convergent (test fails).

If $\sum \frac{1}{n}$ or $\sum \frac{1}{n^2}$

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = 1 \Rightarrow \text{Test FAILS}$$

Use Ratio Test when a_n involves factorials or exponentials

EXAMPLE 7. *Determine if the following series are convergent or divergent:*

(a) $\sum_{n=1}^{\infty} \frac{(-10)^n}{4^{2n+1}(n+1)}$

$$|a_n| = \frac{10^n}{4^{2n+1}(n+1)}$$

$$|a_{n+1}| = \frac{10^{n+1}}{4^{2(n+1)+1}(n+1+1)} = \frac{10^{n+1}}{4^{2n+3}(n+2)}$$

Use Ratio Test

$$L = \lim_{n\to\infty} |a_{n+1}| \cdot \frac{1}{|a_n|} = \frac{10^{n+1}}{4^{2n+3}(n+2)} \cdot \frac{4^{2n+1}(n+1)}{10^n}$$

$$= \lim_{n\to\infty} \frac{\cancel{10^n} \cdot 10 \cdot \cancel{4^{2n+1}}(n+1)}{\cancel{4^{2n+1}} \cdot 4^2 \, (n+2) \cdot \cancel{10^n}} = \frac{5}{8} \lim_{n\to\infty} \frac{n+1}{n+2} = \frac{5}{8} < 1$$

$\Rightarrow$ absolutely convergent $\Rightarrow$ convergent

(b) $\sum_{n=1}^{\infty} \frac{n!}{5^n}$

$$|a_n| = \frac{n!}{5^n} \qquad |a_{n+1}| = \frac{(n+1)!}{5^{n+1}}$$

Note that $n! = 1\cdot2\cdot3\ldots\cdot(n-1)\cdot n$ (Also $0! = 1$,

$$(n+1)! = \underbrace{1\cdot2\cdot3\cdot\ldots(n-1)\cdot n}_{n!}\cdot(n+1) = n!(n+1)$$

Apply Ratio Test

$$L = \lim_{n\to\infty} |a_{n+1}|\cdot\frac{1}{|a_n|} = \lim_{n\to\infty} \frac{(n+1)!}{5^{n+1}}\cdot\frac{5^n}{n!}$$

$$= \lim_{n\to\infty} \frac{\cancel{n!}(n+1)\cdot\cancel{5^n}}{\cancel{5^n}\cdot5\ \ \cancel{n!}} = \lim_{n\to\infty} \frac{n+1}{5} = \infty > 1$$

the series diverges.

(c) $\sum_{n=1}^{\infty} \frac{(2n+1)!}{n!10^n}$

$$|a_n| = \frac{(2n+1)!}{n! \cdot 10^n}$$

$$|a_{n+1}| = \frac{(2(n+1)+1)!}{(n+1)! \cdot 10^{n+1}} = \frac{(2n+3)!}{(n+1)!\, 10^{n+1}}$$

Note $(n+1)! = n!\,(n+1)$

$$\left.\begin{aligned} (2n+1)! &= 1\cdot 2\cdot 3 \cdot \ldots \cdot (2n)\cdot(2n+1) \\ (2n+3)! &= \underbrace{1\cdot 2\cdot 3\cdot \ldots \cdot (2n)(2n+1)}_{(2n+1)!}(2n+2)(2n+3) \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow (2n+3)! = (2n+1)!\,(2n+2)(2n+3)$$

Continue $\longrightarrow$

$$L = \lim_{n \to \infty} |a_{n+1}| \cdot \frac{1}{|a_n|}$$

$$= \lim_{n \to \infty} \frac{(2n+3)!}{(n+1)!\ 10^{n+1}} \cdot \frac{n!\ 10^n}{(2n+1)!}$$

$$= \lim_{n \to \infty} \frac{\cancel{(2n+1)!}\,(2n+2)(2n+3) \cdot \cancel{n!}\ \cancel{10^n}}{\cancel{n!}\,(n+1) \cdot \cancel{10^n} \cdot 10 \cdot \cancel{(2n+1)!}}$$

$$= \lim_{n \to \infty} \frac{(2n+2)(2n+3)}{10(n+1)} = \infty > 1$$

The series diverges.

Flow Chart for Convergence Tests