## 10.7: Taylor and Maclaurin Series

*Problem: Assume that a function $f(x)$ has a power series representation about $x = a$:*

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

*and $f(x)$ has derivatives of every order. Find formula for $c_n$ in terms of $f$.*

*Solution.* We have

$f(x) = C_0 + C_1 (x-a) + C_2 (x-a)^2 + C_3 (x-a)^3 + C_4 (x-a)^4 + \ldots$

Putting $x = a$ in $f(x)$, we get $\boxed{C_0 = f(a)} = \dfrac{f^{(0)}(a)}{0!}$

$f'(x) = 0 + C_1 + 2C_2 (x-a) + 3C_3 (x-a)^2 + 4C_4 (x-a)^3 + \ldots$

Substituting $x = a$ we have $\boxed{C_1 = \dfrac{f'(a)}{1}} = \dfrac{f'(a)}{1!}$

Similarly,

$f''(x) = 2C_2 + 3 \cdot 2 C_3 (x-a) + 4 \cdot 3 \cdot C_4 (x-a)^2 + \ldots$

then $x = a \Rightarrow f''(a) = 2C_2$ and $\boxed{C_2 = \dfrac{f''(a)}{2 \cdot 1}} = \dfrac{f''(a)}{2!}$

$f'''(x) = 3 \cdot 2 C_3 + 4 \cdot 3 \cdot 2 C_4 (x-a) + \ldots$

then $x = a \Rightarrow f'''(a) = 3 \cdot 2 C_3$ and $\boxed{C_3 = \dfrac{f'''(a)}{3 \cdot 2 \cdot 1}} = \dfrac{f'''(a)}{3!}$

Continuing in this manner, you can see the pattern:

$f^{(n)}(a) = n!\, C_n \Rightarrow \boxed{C_n = \dfrac{f^{(n)}(a)}{n!}}$

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

$= c_n$

**The Taylor series for $f(x)$ about $x = a$:**

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n =$$

Quadratic approximation

$$= \overbrace{\underbrace{f(a) + f'(a)(x-a)}_{\text{Linear approximation.}} + \frac{f''(a)}{2!}(x-a)^2}^{\text{Quadratic approximation}} + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

partialsum

Split the Taylor series as follows:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \underbrace{\overbrace{\sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n}^{\text{partialsum}}}_{\substack{T_N(x) \\ N-\text{th degree} \\ \textbf{Taylor polynomial}}} + \underbrace{\sum_{n=N+1}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n}_{\substack{R_N(x) \\ \textbf{Remainder}}}$$

THEOREM 1. *If* $\lim_{n\to\infty} R_N(x) = 0$ *when* $|x-a| < R$ *then*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n, \qquad |x-a| < R.$$

REMARK 2. In all examples that we will be looking at, we assume that $f(x)$ has a power series expansion, i.e.
$\lim_{n\to\infty} R_N(x) = 0$ for some $R$. (This means you don't need to show it.)

EXAMPLE 3. *Given that function $f$ has power series expansion (i.e. Taylor series) centered at 4. Find this expansion and its radius of convergence if it is given that*

$$f^{(n)}(4) = \frac{(-1)^{n-1} 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-3)}{2^{3n-1} n}$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(4)}{n!}(x-4)^n = \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^{n-1} 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-3)}{2^{3n-1} \cdot n \; n!}(x-4)^n}_{\text{Taylor series centered at } x=4}$$

Ratio Test for R:

$$|a_n| = \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-3)\, |x-4|^n}{2^{3n-1}\, n \cdot n!}$$

$$|a_{n+1}| = \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2(n+1)-3)\, |x-4|^{n+1}}{2^{3(n+1)-1} (n+1)\, (n+1)!} = \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot \overset{(2n-3)}{\checkmark} \cdot (2n-1)\, |x-4|^{n+1}}{\underbrace{2^{3n+2}}_{2^{3n-1+3}} (n+1) \underbrace{(n+1)!}_{n!\,(n+1)}}$$

$$L = \lim_{n\to\infty} |a_{n+1}| \cdot \frac{1}{|a_n|} = \lim_{n\to\infty} \frac{\cancel{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-3)}(2n-1)\, |x-4|^{\cancel{n+1}}}{\cancel{2^{3n-1}}\, 2^3\, (n+1)^2\, \cancel{n!}} \cdot \frac{\cancel{2^{3n-1}}\; n \cdot \cancel{n!}}{\cancel{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-3)\, |x-4|^n}}$$

$$= \frac{|x-4|}{8} \underbrace{\lim_{n\to\infty} \frac{(2n-1)\,n}{(n+1)^2}}_{2} = \frac{|x-4|}{8} \cdot 2 = \frac{|x-4|}{4} < 1$$

$$|x-4| < 4$$

$$\boxed{R = 4}$$

EXAMPLE 4. *Find Taylor series for $f(x) = e^{3x}$ at $x = 1$. What is the associated radius of convergence?*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n$$

$$f(x) = e^{3x}$$

$$f'(x) = 3e^{3x}$$

$$f''(x) = 3 \cdot 3e^{3x} = 3^2 e^{3x}$$

$$\vdots$$

$$f^{(n)}(x) = 3^n e^{3x} \Rightarrow f^{(n)}(1) = 3^n e^3$$

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{3^n e^3}{n!}(x-1)^n}$$

Radius of conv. by Ratio Test

$$L = \lim_{n\to\infty} |a_{n+1}| \cdot \frac{1}{|a_n|} = \lim_{n\to\infty} \frac{3^{\cancel{n+1}} \cancel{e^3} |x-1|^{\cancel{n+1}}}{\cancel{(n+1)!} \cdot n+1} \cdot \frac{\cancel{n!}}{\cancel{3^n} \cancel{e^3} \cancel{|x-1|^n}}$$

$$= 3|x-1| \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1 \text{ for all } x$$

$$\boxed{R = \infty}$$

EXAMPLE 5. *Find Taylor series for $f(x) = \ln x$ at $x = 1$. What is the associated radius of convergence?*

| | $x=1$ |
|---|---|
| $f(x) = \ln x$ | $f(1) = 0 = c_0$ |
| $f'(x) = \frac{1}{x}$ | $f'(1) = 1 = 0!$ |
| $f''(x) = -\frac{1}{x^2}$ | $f''(1) = -1 = -1!$ |
| $f'''(x) = \frac{2}{x^3}$ | $f'''(1) = 2 = 2!$ |
| $f^{(4)}(x) = -\frac{2\cdot 3}{x^4}$ | $f^{(4)}(1) = -2\cdot 3 = -3!$ |
| $f^{(5)}(x) = \frac{2\cdot 3\cdot 4}{x^5}$ | $f^{(5)}(1) = 2\cdot 3\cdot 4 = 4!$ |
| | $f^{(6)}(1) = -2\cdot 3\cdot 4\cdot 5 = -5!$ |

$$f^{(n)}(1) = (-1)^{n+1}(n-1)!$$

Note $f^{(n)}(x) = \dfrac{(-1)^{n+1}(n-1)!}{x^n}$

$$c_n = \frac{f^{(n)}(1)}{n!} = \begin{cases} 0 & , n=0 \\ \dfrac{(-1)^{n+1}(n-1)!}{\underbrace{n!}_{(n-1)!\,n}} = \dfrac{(-1)^{n+1}}{n} & , n \geq 1 \end{cases}$$

$$f(x) = \sum_{n=0}^{\infty} c_n (x-1)^n$$

$$\ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n$$

Ratio Test:

$$L = \lim_{n\to\infty} |a_{n+1}| \cdot \frac{1}{|a_n|} = \lim_{n\to\infty} \frac{|x-1|^{n+1}}{n+1} \cdot \frac{n}{|x-1|^n}$$

$$L = |x-1| \lim_{n\to\infty} \frac{n}{n+1} = |x-1| < 1$$

$$\Downarrow$$

$$\boxed{R = 1}$$

EXAMPLE 6. *Find Taylor series for* $\ln(1+x)$ *centered at* $x = 0$. *What is the associated radius of convergence?*

By Example 5 we have

$$\ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n, \quad |x-1|<1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(1+x-1)^n, \quad |1+x-1|<1$$

$$\underbrace{\ln(1+x) = \sum_{n=1}^{\infty} \underbrace{\frac{(-1)^{n+1}}{n}}_{=c_n} x^n}_{\text{Taylor series centered at zero}}, \quad |x|<1 \Rightarrow \boxed{R=1}$$

Taylor series centered at zero = McLaurien series

Question $f(x) = \ln(1+x)$

What is $f^{(100)}(0)$ ?

$$c_n = \frac{f^{(n)}(0)}{n!} \Rightarrow \quad c_{100} = \frac{f^{(100)}(0)}{100!}$$

$$f^{(100)}(0) = 100!\, c_{100} = 100! \frac{(-1)^{100+1}}{100}$$

$$f^{(100)}(0) = -99!$$

**The Maclaurin series is the Taylor series about $x = 0$ (i.e. a=0):**

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots$$

EXAMPLE 7. *Find the Maclaurin series for $f(x)$:*

**(a)** $f(x) = e^x$

$f^{(n)}(x) = e^x \Rightarrow f^{(n)}(0) = e^0 = 1$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$R = \infty$ (Check it!)

**(b)** $f(x) = e^{-x}$

using (a):

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$$

**(c)** $f(x) = x^5 e^{-2x^2}$

Use (a):

$$f(x) = x^5 \cdot \sum_{n=0}^{\infty} \frac{(-2x^2)^n}{n!} = \sum_{n=0}^{\infty} x^5 \frac{(-2)^n x^{2n}}{n!}$$

$$x^5 e^{-2x^2} = \sum_{n=0}^{\infty} \frac{(-2)^n}{n!} x^{2n+5}$$

EXAMPLE 8. *Find the Maclaurin series for $f(x)$:*

**(a)** $f(x) = \cos x = \displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

| | $x=0$ |
|---|---|
| $f(x) = \cos x$ | $1$ |
| $f'(x) = -\sin x$ | $0$ |
| $f''(x) = -\cos x$ | $-1$ |
| $f'''(x) = \sin x$ | $0$ |
| $f^{(4)}(x) = \cos x$ | $1$ |

cycle

odd derivatives

$$f^{(n)}(0) = \begin{cases} 0, & n = 2k+1 \text{ (or } n \text{ is odd)} \\ (-1)^k & n = 2k \text{ (or } n \text{ is even)} \end{cases}$$

$$\boxed{\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}}$$

Use Ratio Test to find radius of convergence

$$L = \lim_{k\to\infty} |a_{k+1}| \cdot \frac{1}{|a_k|} = \lim_{k\to\infty} \frac{|x|^{2(k+1)}}{\underbrace{(2(k+1))!}_{(2k+2)!}} \cdot \frac{(2k)!}{|x|^{2k}}$$

$$= \lim_{k\to\infty} \frac{\cancel{|x|^{2k}}^{+2} \cdot \cancel{(2k)!}}{\cancel{(2k)!}(2k+1)(2k+2)\cancel{|x|^{2k}}} = |x|^2 \lim_{k\to\infty} \frac{1}{(2k+1)(2k+2)} = 0 < 1$$

for all $x$

$$\boxed{R = \infty}$$

(b) $f(x) = \sin x$

Way 1 Use the same method as in item (a)

Way 2 Use termwise integration or differentiation

$$\sin x = \int \cos x \, dx \overset{(a)}{=\!=} \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \int x^{2n} dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \frac{x^{2n+1}}{2n+1} + C$$

To determine $C$ plug in the center of the series: $x=0$

$$\sin 0 = 0 + C \Rightarrow C = 0$$

$$\boxed{\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \quad R = \infty.}$$

Remark Decomposition of $\sin x$ contains only odd polynomials : $x, x^3, x^5, \ldots$

Decomp. of $\cos x$ contains only even polynomials: $1, x^2, x^4, \ldots$

This happen because $\sin x$ is odd function $\sin(-x) = \sin x$

and $\cos x$ is even function $\cos(-x) = -\cos x$.

*Known Mclaurin series and their intervals of convergence you must have memorized:*

geometric

$$
\begin{array}{rcccccl}
\dfrac{1}{1-x} & = & \displaystyle\sum_{n=0}^{\infty} x^n & = & 1 + x + x^2 + x^3 + \dots & & (1,1) \\
e^x & = & \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} & = & 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dots & & (-\infty,\infty) \\
\cos x & = & \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} & = & 1 - \dfrac{x^2}{2} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \dots & & (-\infty,\infty) \\
\sin x & = & \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} & = & x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \dots & & (-\infty,\infty) \\
\arctan x & = & \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} & = & x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \cdots & & [-1,1]
\end{array}
$$

intervals of convergence

very useful for limit calculations and for Alternating Series Remainder estimate

Use the previous table

EXAMPLE 9. *Find the sum of the series:*

**(a)** $$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{3^{2n}(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{\pi}{3}\right)^{2n} = \cos\frac{\pi}{3} = \boxed{\frac{1}{2}}$$

**(b)** $$\sum_{n=0}^{\infty} \frac{2012^n}{n!} = e^{2012}$$

**(c)** $$\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{2n+1}$$

$$= \arctan(x^2)$$

EXAMPLE 10. **(a)** *Determine Maclaurin Series for* $\underbrace{\int \frac{\sin x}{x}\,dx}_{\text{We don't know anti-derivative}}$

Use the table:

$$\frac{\sin x}{x} = \frac{1}{x}\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n}$$

Use termwise integration:

$$\int \frac{\sin x}{x}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \int x^{2n}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot \frac{x^{2n+1}}{2n+1} + C$$

**(b)** *Evaluate* $\int_0^1 \frac{\sin x}{x}\,dx$ *correct to within an error of* $0.001 = \frac{1}{1000}$

In other words, approximate $\int_0^1 \frac{\sin x}{x}\,dx \approx S_n$

such that $|R_n| \leq 0.001$.

$\Downarrow$

find $n$ such that $|R_n| \leq 0.001$.

$$\int_0^1 \frac{\sin x}{x}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!\,(2n+1)} x^{2n+1}\Big|_0^1 = \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!\,(2n+1)}}_{\text{Alternating series}}$$

$$= 1^{2n+1} - 0^{2n+1} = 1$$

$$= 1 - \frac{1}{3!\cdot 3} + \frac{1}{5!\,5} - \frac{1}{7!\,7} + \dots$$

$$\int_0^1 \frac{\sin x}{x}\,dx \approx S_0 = 1 \Rightarrow |R_0| \leq \frac{1}{3!3} = \frac{1}{6\cdot 3} = \frac{1}{18} > 0.001 = \frac{1}{1000}$$

$$\int_0^1 \frac{\sin x}{x}\,dx \approx S_1 = 1 - \frac{1}{3!3} \Rightarrow |R_1| = \frac{1}{5!5} = \frac{1}{120\cdot 5} = \frac{1}{600} > \frac{1}{1000}$$

$$\int_0^1 \frac{\sin x}{x}\,dx \approx S_2 = \boxed{1 - \frac{1}{3!3} + \frac{1}{5!5} \approx 0.9461}$$

$\Downarrow$

$$|R_2| = \frac{1}{7!7} = \frac{1}{35280} < \frac{1}{1000}$$

EXAMPLE 11. *Use series to find the limit:* $\lim_{x\to 0} \dfrac{1-\cos x}{1+x-e^x}$

We know Maclaurin series for $\cos x$ and $e^x$:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\lim_{x\to 0} \frac{1-\cos x}{1+x-e^x} = \lim_{x\to 0} \frac{\cancel{1} - (\cancel{1} - \frac{x^2}{2} + \frac{x^4}{4!} + \dots)}{\cancel{1+x} - (\cancel{1+x} + \frac{x^2}{2} + \frac{x^3}{3!} + \dots)}$$

$$= \lim_{x\to 0} - \frac{\frac{x^2}{2} - \frac{x^4}{4!} + \dots}{\frac{x^2}{2} + \frac{x^3}{3!} + \dots} = \lim_{x\to 0} - \frac{\cancel{x^2}(\frac{1}{2} - \frac{x^2}{4!} + \dots)}{\cancel{x^2}(\frac{1}{2} + \frac{x}{3!} + \dots)}$$

$$= - \frac{\frac{1}{2} - 0 + \dots}{\frac{1}{2} + 0 + 0 + \dots} = \boxed{-1}$$