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CANONICAL FRAMES FOR DISTRIBUTIONS OF ODD RANK AND CORANK 2 WITH MAXIMAL FIRST KRONECKER INDEX

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Abstract. We construct canonical frames and find all maximally symmetric models for a natural generic class of corank 2 distributions on manifolds of odd dimension greater or equal to 7. This class of distributions is characterized by the following two conditions: the pencil of 2-forms associated with the corresponding Pfaffian system has the maximal possible first Kronecker index and the Lie square of the subdistribution generated by the kernels of all these 2-forms is equal to the original distribution. In particular, we show that the unique, up to a local equivalence, maximally symmetric model in this class of distributions with given dimension of the ambient manifold exists if and only if the dimension of the ambient manifold is equal to 7, 9, 11, 15 or $8l-3$, $l \in \mathbb{N}$. Besides, if the dimension of the ambient manifold is equal to 19, then there exist two maximally symmetric models, up to a local equivalence, distinguished by certain discrete invariant. For all other dimensions of ambient manifold there are families of maximally symmetric models, depending on continuous parameters. Our main tool is the so-called symplectification procedure having its origin in Optimal Control Theory. Our results can be seen as an extension of some classical Cartan's results on rank 3 distributions in $\mathbb{R}^5$ to corank 2 distributions of higher odd rank.

1. Introduction

1.1. **Distributions and their Tanaka symbols.** *A distribution D of rank l on a n-dimensional manifold M or an (l,n)-distribution* is a subbundle of the tangent bundle TM with l-dimensional fiber. *The corank of an (l,n)- distribution* by definition is equal to $n-l$. Obviously corank is equal to the number of independent Pfaffian equations defining D. Distributions appear naturally in Control Theory as control systems linear with respect to controls and in the geometric theory of ordinary and partial differential equations.

The general problem is to determine equivalence for germs of these geometric objects with respect to the natural action of the group of germs of diffeomorphisms of M. Except for several cases such as line distributions, corank one distributions and rank $(2,4)$-distributions generic distributions have functional, and, thus, non-trivial differential invariants.

The basic characteristics of a distribution D is its *weak derived flag* and the *Tanaka symbol*. By taking iterative brackets of vector fields tangent to a distribution, one obtains the filtration of the tangent bundle. More precisely, set $D = D^1$ and define recursively $D^j = D^{j-1} + [D, D^{j-1}]$, $j > 1$. The space $D^j(q)$ is called the *jth power of the distribution D at a point q*. Clearly $D^j \subseteq D^{j+1}$. The filtration $\{D^j\}_{j\in\mathbb{N}}$ is called the *weak derived flag* of a distribution and the tuple of dimensions of the subspaces of this filtration at a given point is called the *small growth vector* of the distribution at this point. A point of M is called a *regular point* of a distribution if the small growth vector is constant in a neighborhood of this point.

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Further, Let $\mathfrak{g}^{-1}(q) \stackrel{\text{def}}{=} D(q)$ and $\mathfrak{g}^{-j}(x) \stackrel{\text{def}}{=} D^j(q)/D^{j-1}(q)$ for $j > 1$ If a point q is regular, then the graded space $\mathfrak{m}_q = \sum_{\leq -1} \mathfrak{g}^j(q)$ can be naturally equipped with a structure of a graded nilpotent Lie algebra called a *symbol* of the distribution D at a point q. Indeed, let $\mathfrak{p}_j : D^j(q) \mapsto \mathfrak{g}^{-j}(q)$ be the canonical projection to a factor space. Take $Y_1 \in \mathfrak{g}^{-i}(q)$ and $Y_2 \in \mathfrak{g}^{-j}(q)$. To define the Lie bracket $[Y_1, Y_2]$ take a local section $\widetilde{Y}_1$ of the distribution D^i and a local section $\widetilde{Y}_2$ of the distribution D^j such that $\mathfrak{p}_i\big(\widetilde{Y}_1(q)\big) = Y_1$ and $\mathfrak{p}_j\big(\widetilde{Y}_2(q)\big) = Y_2$. It is clear that $[\widetilde{Y}_1, \widetilde{Y}_2](q) \in D^{i+j}(q)$. Put

$$[Y_1, Y_2] \stackrel{\text{def}}{=} \mathfrak{p}_{i+j}\big([\widetilde{Y}_1, \widetilde{Y}_2](q)\big). \tag{1.1}$$

It is easy to see that the right-hand side of (1.1) does not depend on the choice of sections $\widetilde{Y}_1$ and $\widetilde{Y}_2$. Besides, $\mathfrak{g}^{-1}(q)$ generates the whole algebra $\mathfrak{m}(q)$. A graded Lie algebra satisfying the last property is called *fundamental.*

One can define the *flat distribution* $D_{\mathfrak{m}}$ *of constant fundamental symbol* $\mathfrak{m}$. For this let $M(\mathfrak{m})$ be the connected, simply connected Lie group with the Lie algebra $\mathfrak{m}$ and let e be its identity. Then $D_{\mathfrak{m}}$ is the left invariant distribution on $M(\mathfrak{m})$ such that $D_{\mathfrak{m}}(e) = \mathfrak{g}^{-1}$.

The notion of symbol is extensively used in works of N. Tanaka and his school ([13, 14, 11, 12, 15]) who developed the prolongation procedure to construct canonical frames (coframes) for distributions of so-called constant type, i.e. when the symbols at different points are isomorphic as graded Lie algebras. In particular, as it was proved in [13], for any fundamental symbol $\mathfrak{m}$ the flat distribution $D_{\mathfrak{m}}$ has the algebra of infinitesimal symmetries of maximal dimension among all distributions of constant symbol $\mathfrak{m}$ and this algebra can be described algebraically in terms of the so-called *universal prolongation of the* $\mathfrak{m}$, which is in essence the maximal (non-degenerate) graded Lie algebra, containing the graded Lie algebra $\mathfrak{m}$ as its negative part.

Consider $(2k+1, 2k+3)$-distributions D with small growth vector $(2k+1, 2k+3)$. The case $k = 1$, i.e. the case of $(3, 5)$-distributions, was treated already by Elie Cartan in [1]. Such distributions have the prescribed symbol and the flat distribution is nothing but the Cartan distribution on the space $J^1(\mathbb{R}, \mathbb{R}^2)$ of the 1-jets of functions from $\mathbb{R}$ to $\mathbb{R}^2$. So it has the infinite dimensional group of symmetries. Besides, there exists the unique rank 2 subdistribution $\widetilde{D} \subset D$ such that $\widetilde{D}^2 \subset D$. Moreover, the subdistribution $\widetilde{D}$ is integrable if and only if D is locally equivalent to the flat distribution. If the sub-distribution $\widetilde{D}$ is not integrable, then the germ of $\widetilde{D}$ at some point satisfies $\widetilde{D}^2 = D$ and the small growth vector of $\widetilde{D}$ is $(2, 3, 5)$. So the equivalence problem for D is reduced to the equivalence problem for $\widetilde{D}$. The subdistribution $\widetilde{D}$ has constant symbol and the universal Tanaka prolongation of this symbol is equal to the exceptional Lie algebra G_2.

Now consider the case of an arbitrary k. Obviously, the Lie algebra structure of the symbol $\mathfrak{m}(q) = D(q) \oplus T_qM/D(q)$ is encoded by the map $A_q \in \operatorname{Hom}\big(\bigwedge^2 D(q), T_qM/D(q)\big)$ such that

$$A_q(X, Y) = [X, Y], \quad X, Y \in D(q),$$

where the Lie brackets in the right-hand side are as in the symbol $\mathfrak{m}(q)$. Equivalently, one can consider its dual $A_q^* \in \operatorname{Hom}\big((T_qM/D(q)^*, \bigwedge^2 D(q)^*\big)$,

$$A_q^*(p)(X, Y) = p([X, Y]) \quad X, Y \in D(q),\ p \in \big(T_qM/D(q)\big)^*, \tag{1.2}$$

which can be seen as the pencil of skew-symmetric forms on $D(q)$. This pencil is called the *pencil associated with the distribution D at the point q*. So, all symbols of such distributions are in one-to-one correspondence with the equivalence classes of pencils of skew-symmetric forms on $(2k+1)$-dimensional linear space. The canonical forms of pencils of matrices are given by the classical theorems of Weierstrass and Kronecker (see [7, Chapter 12]). For pencils of skew-symmetric bilinear forms they are specified in [8, Section 6]). With the help of these forms it was shown recently ([2]) that *for any symbol of* $(2k+1, 2k+$

3)$-$*distributions the corresponding flat distribution has an infinite dimensional algebra of infinitesimal symmetries.*

1.2. **Genericity assumptions and description of main results.** On the other hand, by analogy with the case $k=1$ we can define a natural generic subclass of $(2k+1, 2k+3)-$distributions with finite dimensional algebra of infinitesimal symmetries. For this one can distinguish a special subdistribution $\widetilde{D}$ of D (may be with singularities), satisfying $\widetilde{D}^2 \subset D$. The above-mentioned generic subclass of corank 2 distributions will be defined according to the weak derived flag of $\widetilde{D}$. More precisely, let us fix an auxiliary volume form Ω on $D(q)$ and for any $p \in \big(T_qM/D(q)\big)^*$, define a vector $X_p \in D$ via the relation

$$i_{X_p}\Omega = \wedge^k A_q^*(p), \tag{1.3}$$

Then the following subspace $\widetilde{D}(q)$ of $D(q)$

$$\widetilde{D}(q) = \operatorname{span}\{X_p(q) : p \in (T_qM/D(q)^*\} \tag{1.4}$$

is well defined independently of the choice of Ω. The following statement is immediate from (1.3):

Lemma 1.1. *The assignment $p \mapsto X_p$ is a vector-valued degree k homogeneous polynomial on $(T_qM/D(q)^*$ and $\dim \widetilde{D}(q) \le k+1$.*

It is easy to observe from the definition of $\widetilde{D}$ that

$$\widetilde{D}^2 \subset D \tag{1.5}$$

(see also [9, Proposition 2]). Therefore for a flat distribution the subdistribution $\widetilde{D}$ is integrable.

In the present paper we will consider $(2k+1, 2k+3)$-distributions D with $D^2 = TM$, satisfying the following two genericity assumptions

- (G1) $\dim \widetilde{D} \equiv k+1$;
- (G2) $\widetilde{D}^2 = D$.

Note that under condition (G1) the projectivization of the assignment $p \mapsto X_p$ at any point $q \in M$ defines a *rational normal curve* in the projective space $\mathbb{P}(\widetilde{D}(q))$ (or the Veronese embedding of the real projective line $\mathbb{RP}^1$ into $\mathbb{P}(\widetilde{D}(x))$). In particular, for $k=2$ this curve defines the quadric or, equivalently, the sign-indefinite quadratic form Q, up to a multiplication by a nonzero constant on $\widetilde{D}$.

Condition (G1) can be described in terms of the so-called *first minimal index* or the *first Kronecker index* of the pencil associated with D. Since $\dim D(q)$ is odd, this pencil is singular, i.e. each form in it has a nontrivial kernel. Moreover, there exists a homogeneous polynomial map $B : T_qM/D(q) \to D(q)$ such that $B_q(p) \in \ker A_q^*(p)$ and $B_q \neq 0$. The *first minimal index* or the *first Kronecker index* of the pencil associated with distribution at q (and also of the distribution D at q) is by definition the minimal possible degree of such polynomial map.

Lemma 1.1 implies that the first Kronecker index is not greater than k. Further, from the Kronecker canonical form for pencils of skew-symmetric matrices [8, Theorem 6.8] one can get

Proposition 1.1. *The following four conditions are equivalent:*

1. *The distribution D satisfies condition (G1);*
2. *The first Kronecker index of D is equal to k at any point, i.e. it is maximal possible at any point;*
3. *For any $q \in M$ and for any $p \in T_qM/D(q)$, $p \neq 0$, the kernel of the corresponding form $A^*(p)$ is one-dimensional or, equivalently, the kernel is spanned by the vector $X_p(q)$, defined by* (1.3).

(4) *The distribtuion D has constant symbol isomorphic to the following graded nilpotent Lie algebra $\mathfrak{g}^{-1}\oplus\mathfrak{g}^{-2}$, where $\mathfrak{g}^{-1} = \mathrm{span}\{\mathbf{x}_0,\dots,\mathbf{x}_k,\mathbf{y}_1,\dots,\mathbf{y}_k\}$, $\mathfrak{g}^{-2} = \mathrm{span}\{\mathbf{z},\mathbf{n}\}$, and all nonzero products are*

$$[\mathbf{x}_i,\mathbf{y}_{k-i}] = \mathbf{z}, \quad [\mathbf{x}_{i+1},\mathbf{y}_{k-i}] = \mathbf{n}, \quad \forall\, 0 \le i \le k-1. \tag{1.6}$$

The item (2) of the previous proposition explains the terminology used in the title of the paper.

Note also that if (G1) holds and $\widetilde{D}^2$ is strictly contained in D then from item (4) of Proposition 1.1 it follows that $\widetilde{D}^3$ is not contained in D so that in general D is not recovered from $\widetilde{D}$. Therefore if one wants to study D via $\widetilde{D}$ one must to assume (G2).

Can we solve the equivalence problem for the class of distributions, satisfying both (G1) and (G2), in the frame of Tanaka theory, applied for subdistribution $\widetilde{D}$ for $k > 1$? For $k = 2$ the subdistribution $\widetilde{D}$ may have 3 different symbols. These symbols can be characterized as follows: The distribution $\widetilde{D}$ has the distinguished rank 2 subdistribution $\bar{D} \subset \widetilde{D}$, satisfying $\bar{D}^2 \subset \widetilde{D}$. Then, depending on the signature of the restriction of the above mentioned sign-indefinite quadratic form Q to the plane $\bar{D}(q)$, one has 3 symbols: *parabolic, hyperbolic,* or *elliptic*. They are explicitly written in [10] (algebras $m7\_3\_3$ (parabolic case), $m7\_3\_6$ (hyperbolic case), and $m7\_3\_6r$ (elliptic case) in the list there). It can be shown that the flat (5,7)-distribution corresponding to the square of the flat distribution with the symbol $m7\_3\_3$ (i.e. parabolic case) is the unique, up to the local equivalence, maximally symmetric among all (5,7)-distributions satisfying conditions (G1) and (G2): the universal prolongation of $m7\_3\_3$ is 9-dimensional, while the universal prolongations of $m7\_3\_6$ (hyperbolic case), and $m7\_3\_6r$ (elliptic case) are 8-dimensional. The graded Lie algebra symbol $m7\_3\_3$ is described as follows: $m7\_3\_3 = \mathfrak{g}^{-1}\oplus\mathfrak{g}^{-2}\oplus\mathfrak{g}^{-3}$ where $\mathfrak{g}^{-1} = \mathrm{span}\{\mathbf{x}_0,\mathbf{x}_1,\mathbf{x}_2\}$, $\mathfrak{g}^{-2} = \mathrm{span}\{\mathbf{y}_1,\mathbf{y}_2\}$, $\mathfrak{g}^{-3} = \mathrm{span}\{\mathbf{z},\mathbf{n}\}$ and all nonzero products are

$$\begin{aligned} &[\mathbf{x}_i,\mathbf{y}_{2-i}] = \mathbf{z}, \qquad [\mathbf{x}_{i+1},\mathbf{y}_{2-i}] = \mathbf{n}, \qquad i = 0,1;\\ &[\mathbf{x}_0,\mathbf{x}_1] = \mathbf{y}_1, \quad [\mathbf{x}_0,\mathbf{x}_2] = \mathbf{y}_2. \end{aligned} \tag{1.7}$$

However, starting from $k = 3$ the set of symbols of $\widetilde{D}$ depends on continuous parameters. So in order to apply Tanaka's theory to the considered class of distributions one has to classify all this symbols and to generalize this theory to the distribution with non-constant symbol.

Instead, we use the so-called *symplectification of the problem* or the *symplectification procedure*. This procedure was already successfully used for other classes of distributions such as rank 2 and rank 3 distributions of so-called maximal class ([16, 17, 3, 4, 5]). It allows to overcome the dependence on symbol in the construction of the canonical frames.

The important object here is the so-called *annihilator $D^\perp$ of D*, which is the subbundle of the cotangent bundle T^*M with the fibers $D^\perp(q) = \{p \in T^*_qM \mid p(D(q)) = 0\}$. By $\mathbb{P}T^*M$ denote the projectivization $\mathbb{P}T^*M$ of the cotangent bundle T^*M, i.e. the fiber bundle over M with the fiber over q equal to the projectivizations of T^*_qM. In the same way let $\mathbb{P}D^\perp$ be the projectivization of $D^\perp$. For a corank 2 distribution D the bundle $\mathbb{P}D^\perp$ has one-dimensional fibers. Besides, $\mathbb{P}D^\perp$ is foliated by the so-called *abnormal extremals* (the characteristic curves of $\mathbb{P}T^*M$). Thus $\mathbb{P}T^*M$ is equipped with two rank 1 distributions V and C: V is the distribution tangent to the fibers and C is the distribution tangent to the foliation of abnormal extremals. Besides, the rank 2 distribution $V\oplus C$ is bracket generating. So, the distributions V and C define the so-called *pseudo-product structure* on $\mathbb{P}D^\perp$. In this way the equivalence problem for the original distribution is reduced to the equivalence problem for such pseudo-product structures.

In the sequel the subdistribution $\widetilde{D}$ will be denoted by D_{k+1} in order to emphasize its rank. The main results of the paper is the construction of the canonical frame for all

$(2k+1, 2k+3)$-distribution D with $k>1$ satisfying assumptions (G1) and (G2) (Theorem 3.1) and the description of all maximally symmetric models for $k>2$ (Corollary 4.1). In particular, we show that the dimension of the infinitesimal symmetries of such distributions is not greater than $2k+6$ if $k \not\equiv 1 \mod 4$ and $k>2$, it is not greater than $2k+7$ if $k \equiv 1 \mod 4$ and $k>1$, and it is not greater than 9 if $k=2$. The latter case $k=2$ also follows from the analysis of the list of 7-dimensional non-degenerate fundamental symbols in [10], but even in this case our construction of the canonical frame is unified for all $(5,7)$-distributions, satisfying conditions (G1) and (G2), independently of the symbol of the corresponding subdistribution D_3. Note that the normal form for the maximally symmetric $(5,7)$-distribution (as the square of the flat dsitribution with the symbol with the product table as in (1.7)) can be obtained from the analysis of our frame as well, but it is too technical to be included here (the case $k=2$ is exceptional as shown in Corollary 2.1 and it needs a separate analysis, while all $k>2$ can be treated uniformly).

Now let us shortly describe our results from section 4 on the maximally symmetric models in the case $k>2$. All maximally symmetric models are given as the left invariant distributions on Lie groups corresponding to certain bi-graded nilpotent Lie algebras. *The unique, up to a local diffeomorphism, maximally symmetric model exist for $k=3$, $k=4$, $k=6$ and $k \equiv 1 \mod 4$.* Further, if $k=8$ (i.e. the dimension of the ambient manifold is equal to 19), then there exist two maximally symmetric models, up to a local equivalence, distinguished by certain discrete invariant. Finally, for $k=7$ and $k>9$ with $k \not\equiv 1 \mod 4$ there are continuous families of distributions having maximal (i.e. $(2k+6)$-dimensional) algebras of infinitesimal symmetries (for details see Corollary 4.1 below).

Now let us give an explicit description of the maximally symmetric model for all k, when it is unique:

1) The case $k=3$. A $(7,9)$-distribution satisfying conditions (G1) and (G2) with maximal (i.e 12-dimensional) algebra of infinitesimal symmetries is locally equivalent to the square of the flat distribution with the symbol algebra $\mathfrak{m} = \mathfrak{g}^{-1} \oplus \mathfrak{g}^{-2} \oplus \mathfrak{g}^{-3}$ where $\mathfrak{g}^{-1} = \operatorname{span}\{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$, $\mathfrak{g}^{-2} = \operatorname{span}\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$, $\mathfrak{g}^{-3} = \operatorname{span}\{\mathbf{z}, \mathbf{n}\}$ and all nonzero products are

$$
\begin{aligned}
&[\mathbf{x}_i, \mathbf{y}_{3-i}] = (-1)^i \mathbf{z}, \qquad [\mathbf{x}_{i+1}, \mathbf{y}_{3-i}] = (-1)^{i+1}(i+1)\mathbf{n}, \quad i = 0,1,2;\\
&[\mathbf{x}_0, \mathbf{x}_1] = \mathbf{y}_1, \quad [\mathbf{x}_0, \mathbf{x}_2] = \mathbf{y}_2, \quad [\mathbf{x}_0, \mathbf{x}_3] = 3\mathbf{y}_3, \quad [\mathbf{x}_1, \mathbf{x}_2] = -2\mathbf{y}_3.
\end{aligned}
\tag{1.8}
$$

2) The case $k=4$. A $(9,11)$-distribution satisfying conditions (G1) and (G2) with maximal (i.e. 14-dimensional) algebra of infinitesimal symmetries is locally equivalent to the square of the flat distribution with the symbol algebra $\mathfrak{m} = \mathfrak{g}^{-1} \oplus \mathfrak{g}^{-2} \oplus \mathfrak{g}^{-3}$ where $\mathfrak{g}^{-1} = \operatorname{span}\{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$, $\mathfrak{g}^{-2} = \operatorname{span}\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \mathbf{y}_4\}$, $\mathfrak{g}^{-3} = \operatorname{span}\{\mathbf{z}, \mathbf{n}\}$ and all nonzero products are

$$
\begin{aligned}
&[\mathbf{x}_i, \mathbf{y}_{4-i}] = (-1)^i \mathbf{z}, \qquad [\mathbf{x}_{i+1}, \mathbf{y}_{4-i}] = (-1)^{i+1}(i+1)\mathbf{n}, \qquad i = 0,1,2,3;\\
&[\mathbf{x}_0, \mathbf{x}_1] = \mathbf{y}_1, \quad [\mathbf{x}_0, \mathbf{x}_2] = \mathbf{y}_2, \quad [\mathbf{x}_0, \mathbf{x}_3] = -\frac{3}{2}\mathbf{y}_3, \quad [\mathbf{x}_0, \mathbf{x}_4] = -4\mathbf{y}_4,\\
&[\mathbf{x}_1, \mathbf{x}_2] = \frac{5}{2}\mathbf{y}_3, \quad [\mathbf{x}_1, \mathbf{x}_3] = \frac{5}{2}\mathbf{y}_4.
\end{aligned}
\tag{1.9}
$$

3) The case $k=6$. A $(13,15)$-distribution satisfying conditions (G1) and (G2) with maximal (i.e. 18-dimensional) algebra of infinitesimal symmetries is locally equivalent to the square of the flat distribution with the symbol algebra $\mathfrak{m} = \mathfrak{g}^{-1} \oplus \mathfrak{g}^{-2} \oplus \mathfrak{g}^{-3}$ where $\mathfrak{g}^{-1} = \operatorname{span}\{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5, \mathbf{x}_6\}$, $\mathfrak{g}^{-2} = \operatorname{span}\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \mathbf{y}_4, \mathbf{y}_5, \mathbf{y}_6\}$, $\mathfrak{g}^{-3} = \operatorname{span}\{\mathbf{z}, \mathbf{n}\}$ and all nonzero products are

$$
\begin{aligned}
&[\mathbf{x}_i,\mathbf{y}_{6-i}]=(-1)^i\mathbf{z}, \qquad [\mathbf{x}_{i+1},\mathbf{y}_{6-i}]=(-1)^{i+1}(i+1)\mathbf{n}, \qquad i=0,1,2,3,4,5;\\
&[\mathbf{x}_0,\mathbf{x}_1]=-\frac{10}{7}\mathbf{y}_1, \quad [\mathbf{x}_0,\mathbf{x}_2]=-\frac{10}{7}\mathbf{y}_2, \quad [\mathbf{x}_0,\mathbf{x}_3]=-\frac{3}{7}\mathbf{y}_3,\\
&[\mathbf{x}_0,\mathbf{x}_4]=\frac{4}{7}\mathbf{y}_4, \quad [\mathbf{x}_0,\mathbf{x}_5]=\frac{25}{7}\mathbf{y}_5, \quad [\mathbf{x}_0,\mathbf{x}_6]=\frac{60}{7}\mathbf{y}_6,\\
&[\mathbf{x}_1,\mathbf{x}_2]=-\mathbf{y}_3, \quad [\mathbf{x}_1,\mathbf{x}_3]=-\mathbf{y}_4, \quad [\mathbf{x}_1,\mathbf{x}_4]=-3\mathbf{y}_5, \quad [\mathbf{x}_1,\mathbf{x}_5]=-5\mathbf{y}_6,\\
&[\mathbf{x}_2,\mathbf{x}_3]=2\mathbf{y}_5, \quad [\mathbf{x}_2,\mathbf{x}_4]=2\mathbf{y}_6.
\end{aligned}
\tag{1.10}
$$

4) The case $k\equiv 1 \mod 4$. The unique, up to a local equivalence, maximally symmetric models in the case $k\equiv 1 \mod 4$ can be described using the theory of $\mathfrak{sl}_2(\mathbb{R})$ representations. For this let $\mathcal{V}_k$ be the $(k+1)$-dimensional irreducible $\mathfrak{sl}_2(\mathbb{R})$-module, $\mathcal{V}_k=\mathrm{Sym}^k(\mathbb{R}^2)$. Recall that the $\mathfrak{sl}_2(\mathbb{R})$-module $\mathcal{V}_k\otimes\mathcal{V}_l$ with $l\leq k$ decomposes into the irreducible $\mathfrak{sl}_2(\mathbb{R})$ submodules as follows:

$$
\mathcal{V}_k\otimes\mathcal{V}_l\cong\bigoplus_{0\leq s\leq l}\mathcal{V}_{k+l-2s}, \tag{1.11}
$$

while the $\mathfrak{sl}_2(\mathbb{R})$-module $\wedge^2\mathcal{V}_k$ decomposes into the irreducible $\mathfrak{sl}_2(\mathbb{R})$ submodules as follows:

$$
\wedge^2\mathcal{V}_k\cong\bigoplus_{0\leq s\leq\frac{k-1}{2}}\mathcal{V}_{2k-2-4s}, \tag{1.12}
$$

(see, for example,[6]). Let $\sigma_{k,l,s}:\mathcal{V}_k\otimes\mathcal{V}_l\to V_{k+l-2s}$ be the canonical projection w.r.t. the splitting (1.11) and $\tau_{k,s}:\bigwedge^2\mathcal{V}_k\to\mathcal{V}_{2k-2-4s}$ be the canonical projection w.r.t. the splitting (1.12). Note that the k-dimensional subspace appears in the splitting (1.12) if and only if $k\equiv 1 \mod 4$. In this case it corresponds to the index $s=\frac{k-1}{4}$ in the decomposition in the right-hand side of (1.12).

Let $\mathfrak{m}_k=\mathcal{V}_k\oplus\mathcal{V}_{k-1}\oplus\mathcal{V}_1$. Then in the case $k\equiv 1 \mod 4$ the space $\mathfrak{m}_k$ can be equipped with the structure of the graded Lie algebra: First, let $\mathfrak{g}^{-1}=\mathcal{V}_k$, $\mathfrak{g}^{-2}=\mathcal{V}_{k-1}$, $\mathfrak{g}^{-3}=\mathcal{V}_1$. Second, define the Lie product on $\mathfrak{m}_k$ by the following two operators:

$$
\tau_{k,\frac{k-1}{4}}:\wedge^2\mathcal{V}_k\to\mathcal{V}_{k-1}, \quad \sigma_{k,k-1,k-1}:\mathcal{V}_k\otimes\mathcal{V}_{k-1}\to\mathcal{V}_1.
$$

Let us show that this product satisfies the Jacobi identity i.e. that the map $J:\wedge^3\mathcal{V}_k\to\mathcal{V}_1$ defined by

$$
J(v_1,v_2,v_3)=\sum_{\text{cyclic}}\sigma_{k,k-1,k-1}\big(\tau_{k,\frac{k-1}{4}}(v_1,v_2),v_3\big),
$$

is identically equal to zero. First note that by constructions the map J is a homomorphism of $\mathfrak{sl}_2(\mathbb{R})$-modules, i.e. it commutes with the actions of $\mathfrak{sl}_2(\mathbb{R})$ on $\wedge^3\mathcal{V}_k$ and $\mathcal{V}_1$. Assume that J is not identically zero. Then J has to be onto, otherwise its image is a proper $\mathfrak{sl}_2(\mathbb{R})$-submodule of V_1 which is impossible. Therefore the kernel of J is a $\mathfrak{sl}_2(\mathbb{R})$-submodule of $\wedge^3V_k$ of codimension 2. On the other hand, $\wedge^3\mathcal{V}_k$ does not contain such submodule, because the 2-dimensional module $\mathcal{V}_1$ does not appear in the decomposition of $\wedge^3\mathcal{V}_k$ into the irreducible $\mathfrak{sl}_2(\mathbb{R})$-submodules. To prove this recall that the number of appearances of the module $\mathcal{V}_l$ in this decomposition is equal to $N_k(l)-N_k(l+2)$, where

$$
N_k(l)=\#\left\{(i_1,i_2,i_3)\in\mathbb{Z}^3_{\mathrm{odd}}:-k\leq i_1<i_2<i_3\leq k,\ i_1+i_2+i_3=l\right\}
$$

and $\mathbb{Z}_{\mathrm{odd}}$ denotes the set of odd integers. In other words, $N(l)$ is the number of non-ordered triples of pairwise distinct odd integers between $-k$ and k with the sum equal to l. The module $\mathcal{V}_1$ does not appear in this decomposition for $k=2s+1$, $s\in\mathbb{N}$, because in this case $N_k(1) = N_k(3)(= \frac{s(s+1)}{2})$. As a matter of fact, we have proved more general fact that any homomorphism of $\mathfrak{sl}_2(\mathbb{R})$-modules $\wedge^3\mathcal{V}_k$ and $\mathcal{V}_1$ is identically equal to zero.

Further, the square of the flat distribution $\mathfrak{D}_k$ with the symbol algebra $\mathfrak{m}_k$ has $(2k+7)$-dimensional algebra of infinitesimal symmetries isomorphic to the natural semi-direct sum of $\mathfrak{gl}_2(\mathbb{R})$ and $\mathfrak{m}_k$. Indeed, the algebra $\mathfrak{g}^0$ of all derivations of the symbol $\mathfrak{m}_k$ preserving the grading contains the image of the irreducible embedding of $\mathfrak{sl}_2(\mathbb{R})$ into $\mathfrak{gl}(\mathcal{V}_k)$ and the grading element. Therefore $\mathfrak{g}^0$ is at least 4-dimensional and by [13] the algebra of infinitesimal symmetries of the distribution $\mathfrak{D}_k$ (and also of its square) is at least $(2k+7)$-dimensional. On the other hand, by Theorem 3.1 below this algebra is at most $(2k+7)$-dimensional. By Corollary 4.1 the distribution $\mathfrak{D}_k$ is the unique, up to the local equivalence, maximally symmetric model of distributions from the considered class for $k \equiv 1 \mod 4$.

2. Symplectification procedure

2.1. **Characteristic rank 1 distribution on $\mathbb{P}(D^\perp)$.** Let us describe the process of symplectification of the problem. For this first let us recall some standard notions from Symplectic Geometry. Let $\tilde{\pi} : T^*M \mapsto M$ be the canonical projection. For any $\lambda \in T^*M$, $\lambda = (p, q)$, $q \in M$, $p \in T_q^*M$, let $\varsigma(\lambda)(\cdot) = p(\tilde{\pi}_*\cdot)$ be the canonical Liouville form and $\sigma = d\varsigma$ be the standard symplectic structure on T^*M. Given a function $H : T^*M \mapsto \mathbb{R}$ denote by $\vec{H}$ the corresponding Hamiltonian vector field defined by the relation $i_{\vec{H}}\sigma = -dH$. Given a vector field X on M define the function $H_X : T^*M \to \mathbb{R}$, the *quasi-impulse* of X, by $H_X(\lambda) = p\big(X(q)\big)$, where $\lambda = (p, q)$, $q \in M$, $p \in T_q^*M$. The corresponding Hamiltonian vector field $\vec{H}_X$ on T^*M is called the *Hamiltonian lift of the vector field* X. It is easy to show that $\tilde{\pi}_*\vec{H}_X = X$.

As before, let D be a $(2k+1, 2k+3)$-distribution with $D^2 = TM$. If $\mathcal{B}$ is a smooth vector bundle over M, then the sheaf of all smooth sections of $\mathcal{B}$ is denoted by $\Gamma(\mathcal{B})$. For any vector field $Y \in \Gamma(D)$ and any $\lambda = (p, q) \in D^\perp$, where $q \in M$, $p \in T_q^*M$, the vector $\vec{H}_Y(\lambda)$ depends on the vector $Y(q)$ only. This implies that that for any $\lambda \in D^\perp$ we set

$$\vec{H}_D(\lambda) = \operatorname{span}\{\vec{H}_Y(\lambda) : Y \in \Gamma(D)\},$$

then the map $\tilde{\pi}_*|_{\vec{H}_D(\lambda)} : \vec{H}_D(\lambda) \to D\big(\tilde{\pi}(\lambda)\big)$ is an isomorphism. The space $\vec{H}_D(\lambda)$ is called the *Hamiltonian lift of the distribution* D at $\lambda \in D^\perp$.

Further, since $D^\perp$ is an odd dimensional manifold, the restriction $\sigma(\lambda)|_{D^\perp}$ of the standard symplectic form σ on $D^\perp$ has a nontrivial kernel for any $\lambda \in D^\perp$. This kernel can be described in term of the the $\vec{H}_D(\lambda)$. Note that the space $(T_qM/D(q))^*$ is identified canonically with the space $(D^\perp)(q)$. Therefore the map A_q^* from (1.2) can be seen as an element of $\operatorname{Hom}\big((D^\perp)(q), \bigwedge^2 D(q)^*\big)$. Then it is not hard to show that for all $\lambda = (p, q) \in D^\perp$ one has

$$\ker \sigma(\lambda)|_{D^\perp} = \vec{H}_D(\lambda) \cap T_\lambda(D^\perp) = \{v \in \vec{H}_D(\lambda) : \tilde{\pi}_* v \in \ker A_q^*(p)\}. \tag{2.1}$$

Hence from items (2) and (3) of Proposition 1.1 it follows that for corank 2 distributions with maximal first Kronecker index $\ker \sigma(\lambda)|_{D^\perp}$ is one dimensional at any point $p \in D_0^\perp$, where $D_0^\perp$ denotes the annihilator of D without the zero section. In other words, $\ker \sigma|_{D^\perp}$ defines a rank 1 distribution $\widetilde{C}$ on $D_0^\perp$. Besides, from (2.1) it follows that

$$\widetilde{\pi}_*\widetilde{C}(\lambda) = \{\mathbb{R}X_p(q)\}, \quad D_{k+1}(q) = \operatorname{span}\{\widetilde{\pi}_*\big(\widetilde{C}(\lambda)\big) : \lambda \in (D^\perp)_0(q)\}. \tag{2.2}$$

Let, as before, $\mathbb{P}(D^\perp)$ be the projectivization of the annihilator. Since σ is preserved by the flow of the Euler vector field on $D^\perp$, the rank 1 distribution $\widetilde{C}$ on $D_0^\perp$ is well projected to the rank 1 distribution C on $\mathbb{P}(D^\perp)$. This rank 1 distribution is called the *characteristic distribution associated with* D. Integral curves of C are called *characteristic curves* or *abnormal extremals* of D. The reason for the latter name is that by the Pontryagin Maximum Principle in any variational problem on M with non-holonomic constraints defined by D these curves are exactly the extremals with zero Lagrange multiplier near the functional.

2.2. **Canonical filtrations on $T\mathbb{P}(D^\perp)$.** Now let $\pi\colon \mathbb{P}(D^\perp)\to M$ be the canonical projection. Define the lifts of D and D_{k+1} to $\mathbb{P}(D^\perp)$ by the formulae

$$H=\pi_*^{-1}(D),\qquad H_{k+1}=\pi_*^{-1}(D_{k+1}).$$

Note that by constructions the characteristic distribution C is contained in H_{k+1} (see (2.2)). Set

$$V=\pi_*^{-1}(0),$$

In other words, V is the vertical distribution on $\mathbb{P}(D^\perp)$. Note that V has rank 1, because the fibres of $\mathbb{P}(D^\perp)$ are homeomorphic to a circle. By definition of V and relation (1.5) we have

$$[V,H_{k+1}]\subset H_{k+1},\qquad [V,H]\subset H,\qquad [C,H_{k+1}]\subset H. \tag{2.3}$$

An important observation is that for each $\lambda\in\mathbb{P}(D^\perp)$ the spaces $H_{k+1}(\lambda)$ and $H(\lambda)/H_{k+1}(\lambda)$ are equipped with the natural filtrations. The filtration on H_{k+1} is described by the following recursive formula

$$L_0(\lambda)=V(\lambda)\oplus C(\lambda),\quad L_{i+1}(\lambda)=L_i(\lambda)+[V,L_i](\lambda). \tag{2.4}$$

Given a vector v in a linear space denote by $[v]$ its equivalence class in the corresponding projective space. Using (2.2) one gets easily that for any $\lambda=([\bar p],\bar q)\in\mathbb{P}(D^\perp)$ the projectivization of the space π_*L_i coincides with the i-th osculating subspace at the point $[\bar p]$ to the curve $[p]\mapsto[X_p]$, $p\in D_0^\perp(\bar q)$ in $\mathbb{P}(D_{k+1}(q))$. Since the latter is a rational normal curve in $\mathbb{P}(D_{k+1}(q))$, we obtain the following filtration of $H_{k+1}(\lambda)$

$$L_0\subset L_1\subset\cdots\subset L_{k-1}\subset L_k=H_{k+1}, \tag{2.5}$$

where $\dim L_i(\lambda)=i+2$.

Now let us describe the natural filtration on the spaces $H(\lambda)/H_{k+1}(\lambda)$. Recall that there is the canonical quasi-contact distribution Λ on $\mathbb{P}(D^\perp)$ induced by the Liouville form ς on T^*M as follows

$$\Lambda=\mathrm{pr}_*(\ker\varsigma)\subset \mathrm{T\mathbb{P}(D^\perp)},$$

where $\mathrm{pr}\colon \mathrm{D}_0^\perp\to\mathrm{P}(\mathrm{D}^\perp)$ is the quotient mapping. Since $d\varsigma=\sigma$ and $C=\ker\sigma|_{D^\perp}$, the distribution C is the Cauchy characteristic of Λ, i.e. $[C,\Lambda]\subset\Lambda$ and C is the maximal subdistribution with this property. Since by constructions $H\subset\Lambda$, it implies that $[C,H]\subset\Lambda$.

If $h\in\Gamma(C)$ is a locally non-vanishing section of the characteristic distribution C then by (2.3) the Lie brackets $[h,\cdot]$ at λ define the following morphism

$$\mathrm{ad}_h\colon H(\lambda)/H_{k+1}(\lambda)\to\Lambda/H(\lambda). \tag{2.6}$$

First note that this map is onto. Otherwise, $\pi_*\big(C(\lambda)\big)$ is the common kernel for all forms $A^*(p)$, $p\in D^\perp\big(\pi(\lambda)\big)$, of the pencil associated with D at $\pi(\lambda)$, which contradicts the assumption of maximality of the first Kronecker index. Note that $\operatorname{rank} H/H_{k+1}=k$ and $\operatorname{rank}\Lambda/H=1$. Therefore the kernel of $\mathrm{ad}_h\colon H/H_{k+1}\to\Lambda/H$ has rank $k-1$ and it defines a corank one subdistribution $K\subset H$. Note that the morphism in (2.6) is multiplied by a nonzero constant, if one chooses another $h\in\Gamma(C)$. Therefore the distribution K does not depend on this choice.

We have a similar picture on the base manifold M. For any $p\in D^\perp(q)$ consider the morphism $\mathrm{ad}_{X_p}: D(q)\to\ker p$. Then the codimension of $\ker\mathrm{ad}_{X_p}$ in $D(q)$ is equal to 1 and

$$\pi_*K(\lambda)=\ker\mathrm{ad}_{X_p},\quad \lambda=(p,q)\in\mathbb{P}(D^\perp).$$

Further, let $Y_p = (\ker \operatorname{ad}_{X_p})/D_{k+1}(p) \subset D(q)/D_{k+1}(q)$ and

$$Z_p = \{\varphi \in (D(q)/D_{k+1}(q))^* : \varphi(Y_p) = 0\}.$$

Note that $\dim Z_p = 1$. Then, using the normal form of the symbol from the item (4) of Proposition 1.1, it is not hard to get that the assignment $[p] \mapsto [Z_p]$ defines a *rational normal curve* in $\mathbb{P}\big((D(q)/D_{k+1}(q))^*\big)$.

Now let us construct the filtration on K inductively using kernels of natural mappings generated by the iterative brackets with V. Namely, set $K_k = H$, $K_{k-1} = K$, and assume by induction that

$$K_i(\lambda) = \left\{ x \in K_{i+1}(\lambda) : \begin{array}{c} \exists\, X \in \Gamma(K_{i+1}) \text{ with } X(\lambda) = x \\ \text{such that } [V, X](\lambda) \in K_{i+1}(\lambda) \end{array} \right\}, \quad i < k-1 \tag{2.7}$$

By constructions $H_{k+1}(\lambda) \subset K_i(\lambda)$ for any i. Set $F_i(\lambda) = K_i(\lambda)/H_{k+1}(\lambda)$. It turns out that the $\pi_* K_i(\lambda)/D_{k+1}(q)$ can be described in terms of the $(k-1-i)$-th osculating space of the curve $[p] \mapsto [Z_p]$. Namely, $\pi_* K_i(\lambda)/D_{k+1}(q)$ is exactly the space of all vectors, annihilated by all elements of these osculating space (recall that the latter space belong to $\mathbb{P}\big((D(q)/D_{k+1}(q))^*\big)$). Since the curve $[p] \mapsto [Z_p]$ is the rational normal curve, we get that the flag $\{F_i(\lambda)\}_{i=1}^{k-1}$ is complete, i.e.

$$0 \subset F_1(\lambda) \subset \cdots \subset F_{k-1}(\lambda) \subset F_k = H(\lambda)/H_{k+1}(\lambda), \quad \dim F_i(\lambda) = i. \tag{2.8}$$

To summarize, filtrations (2.5) and (2.8) are obtained with the help of osculating subspaces to two rational normal curves: $[p] \to [X_p]$ in $\mathbb{P}D_{k+1}$ and $[p] \to [Z_p]$ in $\mathbb{P}\big((D(q)/D_{k+1}(q))^*\big)$.

Till now we used assumption (G1) but not (G2). Now we will assume the following condition weaker than (G_2): the distribution D_{k+1} is not integrable. From this we can extract an additional information from filtrations (2.5) and (2.8) in the form of certain integer-valued invariants, which will be important in the sequel. First let

$$\mathcal{A}_r(\lambda) = H_{k+1}(\lambda) + \operatorname{span}\{[L_s, L_t](\lambda) : s + t \le r, 0 \le s, t \le k\} \tag{2.9}$$

Obviously, $\mathcal{A}_r(\lambda) \subseteq \mathcal{A}_{r+1}(\lambda)$. Since D_{k+1} is not integrable there exists an integer r, $1 \le r \le 2k-1$ such that

$$\mathcal{A}_r(\lambda) \neq H_{k+1}(\lambda).$$

Let

$$w(\lambda) = \min\{r \mid \mathcal{A}_r(\lambda) \neq H_{k+1}(\lambda)\},$$

and

$$i(\lambda) = \min\{i \mid \mathcal{A}_{w(\lambda)} \subset K_i(\lambda)\}.$$

Given $q \in M$ let

$$w_D(q) = \min\{w(\lambda) \mid \lambda \in \mathbb{P}(D^\perp)(q)\}$$

and

$$i_D(q) = \max\{i(\lambda) \mid \lambda \in \mathbb{P}(D^\perp)(q)\}$$

The numbers $w_D(q)$ and $i_D(q)$ are integer-valued invariants of the distribution D at q. A point $q \in M$ is said to be *regular* if w_D and i_D are constant in a neighborhood of q. By constructions, the function $w(\lambda)$ is upper semicontinuous and the function $i(\lambda)$ is lower semicontinuous. It implies that the set of regular points is open and dense subset of M. Also let $\mathcal{R}_1 = \{\lambda \in \mathbb{P}(D^\perp) : w(\lambda) = w_D\big(\pi(\lambda)\big), i(\lambda) = i_D\big(\pi(\lambda)\big)\}$. Then the intersection of $\mathcal{R}_1$ with any fiber of $\mathbb{P}(D^\perp)$ is open set in the Zariski topology of this fiber.

We list several properties of the numbers w_D and i_D.

Lemma 2.1. *The number w_D is odd.*

Proof. Let ε be a section of V and h be a section of C. By (2.4) and (2.5), the subspaces L_i are spanned by vector fields $\varepsilon, h, \mathrm{ad}_\varepsilon h, \ldots, \mathrm{ad}_\varepsilon^i h$. Assume that $[\mathrm{ad}_\varepsilon^s h, \mathrm{ad}_\varepsilon^{w-s-1}h] \in H_{k+1}$ for any $s = 0, \ldots, \lfloor \frac{w-1}{2} \rfloor$ on an open set of the fiber of $\mathbb{P}(D^\perp)$. Applying ad_ε and using the Jacobi identity we get

$$[\mathrm{ad}_\varepsilon^{s+1}h, \mathrm{ad}_\varepsilon^{w-s-1}h] + [\mathrm{ad}_\varepsilon^s h, \mathrm{ad}_\varepsilon^{w-s}h] \in H_{k+1} \tag{2.10}$$

Assume that w is even. Then substituting $s = \frac{w}{2} - 1$ into (2.10), we get that $[\mathrm{ad}_\varepsilon^{\frac{w}{2}-1}h, \mathrm{ad}_\varepsilon^{\frac{w}{2}+1}h] \in H_{k+1}$. Then using (2.10) consecutively we get $[\mathrm{ad}_\varepsilon^s h, \mathrm{ad}_\varepsilon^{w-s}h] = 0 \mod H_{k+1}$ for any $s = 0, \ldots, \frac{w}{2}$. Therefore $w < w_D$ and thus w_D can not be even. □

Remark 2.1. Note that from the similar arguments as in the previous lemma one can show that for every section ε of V and h of C we have

$$[h, \mathrm{ad}_\varepsilon^{w_D}h](\lambda) \equiv (-1)^s[\mathrm{ad}_\varepsilon^s h, \mathrm{ad}_\varepsilon^{w_D-s}h](\lambda) \,\mathrm{mod} H_{k+1}(\lambda), \quad \lambda \in \mathcal{R}_1.$$

This implies that

$$\dim A_{w_D}(\lambda)/H_{k+1}(\lambda) = 1, \quad \lambda \in \mathcal{R}_1 \tag{2.11}$$

□

Lemma 2.2. *If $i_D = 1$ then $[D_{k+1}, D_{k+1}] = D$.*

Proof. If $i_D = 1$ then $K_1(\lambda) \subset [H_{k+1}, H_{k+1}](\lambda)$ for every $\lambda \in P(D^\perp)$. It follows from the constructions that

$$\mathrm{span}\Big\{ \bigcup_{\lambda \in \mathbb{P}(D^\perp)\left(\pi(\lambda)\right)} \pi_*(K_1(\lambda)) \Big\} = D(\pi(\lambda))$$

(as a matter of fact, the curve $[p] \mapsto [\pi_*(K_1(p,q))/D_{k+1}(q)]$, $p \in D^\perp(q)$, is a rational normal curve in $\mathbb{P}\big(D(q)/D_{k+1}(q)\big)$). Hence $[D_{k+1}, D_{k+1}] = D$. □

Lemma 2.3. *If $w_D = 1$ then $i_D \neq k-1$.*

Proof. Let ε be a section of V and h be a section of C. Denote

$$\varepsilon_1 = [h, \varepsilon], \qquad \varepsilon_2 = [h, \varepsilon_1].$$

First of all notice that ε_2 is a section of $H \setminus H_{k+1}$ since $w_D = 1$. By (2.3) $[\varepsilon, \varepsilon_2] \in \Gamma(H)$. Our aim is to prove that $[\varepsilon, \varepsilon_2] \in K$ provided that $\varepsilon_2 \in K$. By definition of the spaces K_i, this statement implies that $i_D \neq k-1$). Assume that $\varepsilon_2 \in K$ and let $Y = [h, [\varepsilon, \varepsilon_2]]$. We will prove that $Y \in \Gamma(H)$.

If $Y \notin \Gamma(H)$ then Y spans Λ modulo H. We will show that it leads to the contradiction. The Jacobi identity and (2.3) imply that

$$[\varepsilon_1, \varepsilon_2] = [h, [\varepsilon, \varepsilon_2]] = Y \mod H,$$

since $[h, e_2] = 0 \mod H$. Moreover we have

$$[\varepsilon_1, [\varepsilon, \varepsilon_2]] = [[h, \varepsilon], [\varepsilon, \varepsilon_2]] = -[\varepsilon, Y] \mod \Lambda$$

and from above we obtain

$$[\varepsilon_2, [\varepsilon, \varepsilon_1]] = [\varepsilon_1, [\varepsilon, \varepsilon_2]] - [\varepsilon, [\varepsilon_1, \varepsilon_2]] = -2[\varepsilon, Y] \mod \Lambda.$$

We use the Jaccobi identity once again and we get

$$[h, [\varepsilon_1, [\varepsilon, \varepsilon_1]]] = [\varepsilon_2, [\varepsilon, \varepsilon_1]] + [\varepsilon_1, [\varepsilon, \varepsilon_2]] = -3[\varepsilon, Y] \mod \Lambda.$$

On the other hand $[\varepsilon_1, [\varepsilon, \varepsilon_1]] \in \Gamma(H)$ and therefore $[h, [\varepsilon_1, [\varepsilon, \varepsilon_1]]] = cY$ for some function c. We conclude that

$$[\varepsilon, Y] = 0 \mod \Lambda.$$

But it means ε is a Cauchy characteristic vector field of Λ, i.e. $[\varepsilon, \Lambda] \subset \Lambda$. It implies that ε is a section of C, which is not the case. Thus we get the contradiction and the proof is completed. □

Corollary 2.1. *If D is a $(5,7)$-distribution with maximal Kronecker index then $i_D = 2$ or D_3 is integrable.*

Proof. Let $h, \varepsilon, \varepsilon_1, \varepsilon_2$ be as in the proof of Lemma 2.3. First let us prove that if D_3 is not integrable then $w_D > 1$ then D_3 is integrable. Assume the converse, i.e. that

$$[h, \varepsilon_1] \in \Gamma(H_3). \tag{2.12}$$

Applying ad_ε to the last relation and using Jacobi identity, we get that

$$[h, [\varepsilon, \varepsilon_1]] \in \Gamma(H_3). \tag{2.13}$$

Applying ad_ε and the Jacobi identity once more we get that

$$-[\varepsilon_1, [\varepsilon, \varepsilon_1]] + [h, [\varepsilon, [\varepsilon, \varepsilon_1]]] \in \Gamma(H_3) \tag{2.14}$$

On the other hand, $[\varepsilon, [\varepsilon, \varepsilon_1]] \in H_3 = \mathrm{span}\{h, \varepsilon, \varepsilon_1, [\varepsilon, \varepsilon_1]\}$, which together with (2.12) and (2.13) implies that $[h, [\varepsilon, [\varepsilon, \varepsilon_1]]] \in \Gamma(H_3)$. Hence (2.14) implies that $[\varepsilon_1, [\varepsilon, \varepsilon_1]] \in \Gamma(H_3)$. So, H_3 is integrable and therefore D_3 is integrable. We get the contradiction.

Thus if D_3 is not integrable, then $w_D = 1$. Since $k = 2$, by the previous lemma $i_D \neq 1$ therefore i_D has to be equal to 2. This completes the proof of the lemma. □

Further let $\widetilde{\mathcal{R}_2}$ be a subset of $\mathcal{R}_1$ consisting of all points λ such that for any r the dimension of subspaces A_r is constant in a neighborhood of λ (in $\mathcal{R}_1$). Obviously, $\widetilde{\mathcal{R}_2}$ is an open and dense subset of $\mathbb{P}(D^\perp)$. Now we will assume that the condition (G2) holds, i.e. $D^2_{k+1} = D$. Then for any λ there exists an integer r such that $K_1(\lambda) \subset \mathcal{A}_r(\lambda)$. Clearly $r(\lambda) \geq w(\lambda)$. Let

$$r(\lambda) = \min\{r : K_1(\lambda) \subset \mathcal{A}_r(\lambda)\}. \tag{2.15}$$

Note that the function $r(\lambda)$ is lower semicontinuous on the set $\widetilde{\mathcal{R}_2}$. Therefore, if $\mathcal{R}_2$ denotes a subset of $\widetilde{\mathcal{R}_2}$ consisting of all points λ such that $r(\lambda)$ is constant in a neighborhood of λ, then $\mathcal{R}_2$ is open and dense in $\mathbb{P}(D^\perp)$. Note that the intersection of $\mathcal{R}_2$ with any fiber of $\mathbb{P}(D^\perp)(q)$ for any $q \in \pi(\mathcal{R}_2)$ is an open set in the Zariski topology of this fiber.

3. Construction of canonical frames

Now we formulate and prove our main result on the frames for distributions from the considered class.

Theorem 3.1. *Assume that a $(2k+1, 2k+3)$-distribution D with $k > 1$ has the maximal first Kronecker index and the square of the subdistribution D_{k+1} is equal to the distribution D. Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be the open dense subsets of $\mathbb{P}(D^\perp)$ defined in the previous sections.*

1. *If w_D is not equal to $\frac{k+1}{2}$, and $i_D \equiv 1$, then there exists a canonical frame on rank 2 bundle over $\mathcal{R}_1$;*
2. *If $w_D \equiv \frac{k+1}{2}$ and $i_D \equiv 1$, then there exists a canonical frame on rank 3 bundle over $\mathcal{R}_1$;*
3. *If i_D is greater than 1 then there exists a canonical frame on rank 1 bundle over a neighborhood of any point of $\mathcal{R}_2$.*

Two corank 2 distributions D and D' satisfying conditions (G1) and (G2) are equivalent if and only if there is a diffeomorphism (of the corresponding bundles) sending the canonical frame of D to the canonical frame of D'.

Note that Lemma 2.1 implies that $w_D = \frac{k+1}{2}$ does not hold unless $k \equiv 1 \mod 4$. If we take into account Corollary 2.1, then Theorem 3.1 implies immediately the following

Corollary 3.1. *Assume that a $(2k+1, 2k+3)$-distribution D has the maximal first Kronecker index and the square of the subdistribution D_{k+1} is equal to the distribution D. Then the dimension of its algebra of infinitesimal symmetries does not exceed*

(1) $2k+6$, *if* $k \not\equiv 1 \mod 4$ *and* $k>2$;
(2) $2k+7$, *if* $k \equiv 1 \mod 4$ *and* $k>1$;
(3) 9, *if* $k=2$.

In section 4 we show that the upper bounds for the algebra of infinitesimal symmetries from the previous Corollary are sharp and describe all corank 2 distributions from the considered class for which these upper bounds are attained.

Proof of Theorem 3.1 Recall that V and C are rank 1 distributions on $\mathbb{P}(D^\perp)$. Let V_0 and C_0 denote the corresponding bundles with zero section removed. Obviously, they are principal $\mathbb{R}^*$-bundles, where $\mathbb{R}^*$ is the multiplicative group of real numbers. Further, recall that the fiber $D^\perp(q)$ of $D^\perp$ over a point $q \in M$ is a plane and the fiber of $\mathbb{P}(D^\perp(q))$ is a projective line. Fix a point $\lambda = (p,q) \in \mathbb{P}(D^\perp)$ and consider all homogeneous coordinates $[x_1 : x_2]$ on $\mathbb{P}(D^\perp(q))$ such that the point $[p]$ is equal to $[1:0]$ in these coordinates. Let B denote the rank 2 bundle over $\mathbb{P}(D^\perp)$ with the fiber over λ consisting of all such homogeneous coordinates on $\mathbb{P}(D^\perp(q))$. In other words, the fiber of B over $\lambda = (p,q)$ is the set of all projective mappings from $\mathbb{RP}^1$ to $\mathbb{P}(D^\perp(q)))$ sending the point $[1:0]$ to $[p]$. Obviously, B is a principal $ST(2,\mathbb{R})$-bundle, where $ST(2,\mathbb{R})$ is the group of 2×2 upper triangular matrices with the determinant 1. The following 2 bundles over $\mathbb{P}(D^\perp)$ play an important role in the sequel

$$B_1 = V_0 \times C_0, \qquad B_2 = B \times C_0.$$

Here B_1 is the bundle over $\mathbb{P}(D^\perp)$ with the fibers equal to the Cartesian product of the corresponding fibers of V_0 and C_0; the bundle B_2 is understood similarly. Obviously, B_1 is a principal $\mathbb{R}^* \times \mathbb{R}^*$-bundle, and B_2 is a principal $T(2,\mathbb{R})$-bundle, where $T(2,\mathbb{R})$ is the group of 2×2 upper triangular matrices.

The group actions define fundamental vector fields on bundles B_1 and B_2. Let us choose bases in the space of fundamental vector fields as follows.

First, let $\mathbf{b}$ denote the vector field on C_0 generating the flow $(\lambda,h) \mapsto (\lambda, e^s h)$, for any $(\lambda,h) \in C_0$, where $\lambda \in \mathbb{P}(D^\perp)$, $h \in C_0(\lambda)$. Since the fibers of C_0 appear as factors for the fibers of B_1 and B_2 we can define the analogous vector field on these bundles as well (just by defining the corresponding flow such that it acts trivially on the remaining factors). We will denote this vector field on B_1 and B_2 by $\mathbf{b}$ as well.

Secondly, let $\mathbf{a}$ denote the vector field on B_1 generating the flow

$$\big(\lambda, (\varepsilon, h)\big) \mapsto \big(\lambda, (e^s\varepsilon, h)\big)$$

for any $\big(\lambda,(\varepsilon,h)\big) \in B_1$, where $\lambda \in \mathbb{P}(D^\perp)$, $\varepsilon \in V_0(\lambda)$, $h \in C_0(\lambda)$. By the same letter $\mathbf{a}$ denote the vector field on B_2 generating the flow

$$\big(\lambda, ([x_1:x_2], h)\big) \mapsto \big(\lambda, ([x_1 : e^{-s}x_2], h)\big),$$

where $\lambda = (p,q) \in \mathbb{P}(D^\perp)$, $[x_1:x_2]$ are homogeneous coordinates on $\mathbb{P}(D^\perp(q))$ such that $[p] = [x_1 : 0]$, and $h \in C_0(\lambda)$.

Finally, let $\mathbf{c}$ denote the vector field on B_2 generating the flow

$$\big(\lambda, ([x_1:x_2], h)\big) \mapsto \big(\lambda, ([x_1 - sx_2 : x_2], h)\big),$$

where λ, $[x_1:x_2]$, and h are as above.

It is easy to show that we have the following relations on B_2

$$[\mathbf{a},\mathbf{b}] = 0, \quad [\mathbf{c},\mathbf{b}] = 0, \quad [\mathbf{a},\mathbf{c}] = -\mathbf{c}, \tag{3.1}$$

and the first relation holds on B_1 as well.

Let $\Pi_i : B_i \to \mathbb{P}(D^\perp)$ be the canonical projection. We say that a vector field E on the bundle B_1 is a *lift of the distribution* V *to* B_1, if for any $\mu_1 = \big(\lambda,(\varepsilon,h)\big) \in B_1$, where

$\lambda \in \mathbb{P}(D^\perp)$, $\varepsilon \in V_0(\lambda)$, $h \in C_0(\lambda)$, one has

$$(\Pi_1)_* E(\mu_1) = \varepsilon.$$

To define the analogous notion on the bundle B_2 first define it on the bundle B. Take $\mu = \big(\lambda, [x_1 : x_2]\big) \in B$ where $\lambda = (p, q) \in \mathbb{P}(D^\perp)$ and $[x_1 : x_2]$ are homogeneous coordinates on $\mathbb{P}(D^\perp(q))$ such that $[p] = [x_1 : 0]$. Then $t = \frac{x_2}{x_1}$ defines coordinates on $\mathbb{P}(D^\perp(q))$ and a *lift of the distribution V to the bundle B* is a vector field E on B, satisfying the following relation for any such μ:

$$(\Pi)_* E(\mu) = \frac{\partial}{\partial t}([p]),$$

where $\Pi : B \to \mathbb{P}(D^\perp)$ is the canonical projection. Finally a *lift of the distribution V to the bundle B_2* is a vector field on B_2 such that $(\mathfrak{P})_* E$ is a lift of V to B, where $\mathfrak{P} : B_2 \to B$ is the canonical projection.

To define the lift of the distribution C to the bundle B_i first define the lift of C to C_0: it is a vector field $\mathfrak{H}$ on C_0 such that if $\widetilde{\Pi} : C_0 \mapsto \mathbb{P}(D^\perp)$, then for any $(\lambda, h) \in C_0$, where $\lambda \in \mathbb{P}(D^\perp)$, $h \in C_0(\lambda)$, one has $(\widetilde{\Pi})_* \mathfrak{H}\big((\lambda, h)\big) = h$. Then the vector field $\mathfrak{H}$ on the bundle B_i, $i = 1, 2$ is called a *lift of the distribution C to B_i* if $(\mathfrak{P}_i)_* \mathfrak{H}$ is a lift of C to C_0, where $\mathfrak{P}_i : B_i \to C_0$ is the canonical projection.

Now let W_i, $i = 1, 2$ be the distribution of tangent spaces to the fibers of B_i, i.e.

$$W_i := \ker(\Pi_i)_*.$$

Distributions W_i are also called the *vertical distribution on B_i*. Lifts E and $\mathfrak{H}$ are defined modulo vertical distributions W_i. By constructions, all lifts E and $\mathfrak{H}$ of V and C, respectively, satisfy the following relations

$$\begin{aligned} &[\mathbf{a}, E] = E \mod W_i, \quad [\mathbf{b}, E] \in \Gamma(W_i), \quad [\mathbf{c}, E] \in \Gamma(W_2), \\ &[\mathbf{b}, \mathfrak{H}] = \mathfrak{H} \mod W_i, \quad [\mathbf{a}, \mathfrak{H}] \in \Gamma(W_i), \quad [\mathbf{c}, \mathfrak{H}] \in \Gamma(W_2) \end{aligned} \tag{3.2}$$

Here the formulas containing $\mathbf{c}$ are related to the bundle B_2 only.

Our goal is to choose the lifts E and $\mathfrak{H}$ in a canonical way. Once it is done one can complete the tuple consisting of the fundamental vertical vector fields and the canonical lifts to the canonical frame on the corresponding bundle B_i by taking appropriate iterative Lie brackets of these canonical lifts.

In the sequel $\widehat{V}$, $\widehat{L}_j$, $\widehat{\mathfrak{H}}_{k+1}$, $\widehat{K}_j$, $\widehat{\mathcal{A}}_j$, $\widehat{H}$, and $\widehat{\Lambda}$ denote the pull backs of distributions V, L_j, $H_k + 1$, K_j, $\mathcal{A}_j$, H, and Λ, respectively, to the corresponding B_i, $i = 1, 2$.

Step 1. The canonical lift of V. First we will work on the bundle B_1. Let E be a lift of V and $\mathfrak{H}$ be a lift of C to B_1. By constructions, vector fields $E, \mathfrak{H}, \mathrm{ad}_E \mathfrak{H}, \ldots, \mathrm{ad}_E^i \mathfrak{H}$ span $\widehat{L}_i$ modulo W_1 and $\widehat{L}_k = \widehat{H}_{k+1}$. It implies that $\mathrm{ad}_E^{k+1} \mathfrak{H} \in \Gamma(\widehat{H}_{k+1})$. Therefore there exist a function η such that

$$\mathrm{ad}_E^{k+1} \mathfrak{H} \equiv \eta \, \mathrm{ad}_E^k \mathfrak{H} \mod \widehat{L}_{k-1} \tag{3.3}$$

First, we are looking for a pair of lifts E and $\mathfrak{H}$ satisfying the condition

$$\mathrm{ad}_E^{k+1} \mathfrak{H} \equiv 0 \mod \widehat{L}_{k-1} \tag{3.4}$$

For this start with some lift E of V and $\mathfrak{H}$ of C and assume that they satisfy (3.3) for some function η. Take other lifts $\widetilde{E}$ and $\widetilde{\mathfrak{H}}$. Then there exist functions α, β,γ, δ such that

$$\widetilde{E} = E + \alpha \mathbf{a} + \beta \mathbf{b}, \quad \widetilde{\mathfrak{H}} = \mathfrak{H} + \gamma \mathbf{a} + \delta \mathbf{b}. \tag{3.5}$$

By direct computations, using relations (3.2), one gets

$$\mathrm{ad}_{\widetilde{E}}^{k+1} \widetilde{\mathfrak{H}} \equiv \big(\eta + (k+1)(\frac{k}{2}\alpha + \beta)\big) \mathrm{ad}_{\widetilde{E}}^k \widetilde{\mathfrak{H}} \mod \widehat{L}_{k-1} \tag{3.6}$$

Thus, a pair of lifts $\widetilde{E}$ and $\widetilde{\mathfrak{H}}$ satisfies condition (3.4) if and only if

$$\frac{k}{2}\alpha + \beta = -\frac{\eta}{k+1}. \tag{3.7}$$

Further, from Remark 2.1 it follows that $[L_0, L_{w_D}](\lambda) \not\subseteq H_{k+1}(\lambda)$ for $\lambda \in \mathcal{R}_1$. Hence there is a section G of $\widehat{K}_1$ on $\Pi_1^{-1}(\mathcal{R}_1)$, unique modulo $\widehat{H}_{k+1}$, such that for any lifts E and $\mathfrak{H}$ of V and C one has

$$\mathrm{ad}_E^{i_D-1} G \equiv [\mathfrak{H}, \mathrm{ad}_E^{w_D}\mathfrak{H}] \quad \mathrm{mod}\ \widehat{K}_{w_D-1}. \tag{3.8}$$

Now assume that $\mu, \tilde{\mu} \in \Pi_1^{-1}(\mathcal{R}_1)$,

$$\mu = \big(\lambda, (\varepsilon, h)\big), \quad \tilde{\mu} = \big(\lambda, (a\varepsilon, bh)\big), \tag{3.9}$$

where $\varepsilon \in V_0(\lambda)$, $h \in C_0(\lambda)$, and $a, b \in \mathbb{R}^*$. Then from (3.8) it follows immediately that

$$(\Pi_1)_* G(\widetilde{\mu}) \equiv a^{w_D - i_D + 1} b^2 (\Pi_1)_* G(\mu) \quad \mathrm{mod}\ \widehat{H}_{k+1}. \tag{3.10}$$

Assume that $\lambda \in \mathcal{R}_2$ and $r(\lambda) \equiv r$ in a neighborhood $\widetilde{U}$ of λ. Choose a local basis of $\widehat{\mathcal{A}}_{r-1}$ in Π_1^{-1} and complete it to a local basis of $\widehat{\mathcal{A}}_r$ by a tuple of vector fields $\big\{[\mathrm{ad}_E^s\mathfrak{H}, \mathrm{ad}_E^{r-s}\mathfrak{H}]\big\}_{s\in\mathcal{S}}$, where $\mathcal{S} \subset \{0, \dots, r\}$. Since by (2.15) G is a section of $\widehat{\mathcal{A}}_r$ but does not belong to $\widehat{\mathcal{A}}_{r-1}$, there exists $\bar{s} \in \mathcal{S}$ such that the coefficient $c_{\bar{s}}$ near one of the field $[\mathrm{ad}_E^{\bar{s}}\mathfrak{H}, \mathrm{ad}_E^{r-\bar{s}}\mathfrak{H}]$ in the expansion of G in the chosen basis does not vanish at any point of $\mathcal{R}_2$ over a neighborhood $U \subset \widetilde{U}$ of λ. Let $\mathcal{U} = \pi^{-1}(U)$. If points $\mu, \tilde{\mu} \in \mathcal{U}$ are related as in (3.9), then using (3.10) it is easy to see that

$$c_{\bar{s}}(\tilde{\mu}) = a^{r - w_D + i_D - 1} c_{\bar{s}}(\mu) \tag{3.11}$$

Note that by constructions $r \geq w_D$. So, if $i_D > 1$ then the power of a in the transformation rule (3.11) is positive. So, we can distinguish the codimension 1 submanifold B_3 of $\Pi_1^{-1}(\mathcal{U})$, consisting of all points of $\Pi_1^{-1}(\mathcal{U})$ with $c_{\bar{s}} = 1$ if $r - w_D + i_D$ is even and with $|c_{\bar{s}}| = 1$ if $r - w_D + i_D$ is odd. As a matter of fact B_3 is a R^*-bundle over $\mathcal{U}$, which is a reduction of B_1. One can naturally identify B_3 with C_0 (over $\mathcal{U}$). Now we can consider only lifts of V and C which are tangent to B_3 or shortly lifts of V and C to B_3. If E and $\widetilde{E}$ are lifts of V to B_3 and $\mathfrak{H}$ and $\widetilde{\mathfrak{H}}$ are lifts of C to B_3 then instead of transformation rule (3.5) we have

$$\widetilde{E} = E + \beta\mathbf{b}, \quad \widetilde{\mathfrak{H}} = \mathfrak{H} + \delta\mathbf{b}. \tag{3.12}$$

So, the normalization condition (3.7) transforms to the condition $\beta = -\frac{\eta}{k+1}$ and gives the canonical lift of V to B_3.

On the other hand, if $i_D = 1$ then by definitions $r = w_D$. Therefore from (3.11) it follows that $c_{\bar{s}}$ is constant on the fibers of B_1 (actually it is identically equal to 1) and we cannot make the above reduction of the bundle B_1. Instead, we are looking for an additional condition for the lifts to B_1. Again fix some lift E and $\mathfrak{H}$ to B_1 of V and C respectively and G is a vector field defined by (3.8) modulo $\widehat{H}_{k+1}$. By constructions, $\widehat{K}_i = \widehat{H}_{k+1} + \mathrm{span}\{G, \mathrm{ad}_E G, \dots, \mathrm{ad}_E^{i-1} G\}$ and and $\widehat{K}_k = \widehat{H}$. It implies that $\mathrm{ad}_E^k G \in \Gamma(\widehat{H})$. Therefore there exist a function υ such that

$$\mathrm{ad}_E^k G \equiv \upsilon\, \mathrm{ad}_E^{k-1} G \quad \mathrm{mod}\, \widehat{K}_{k-1} \tag{3.13}$$

We are looking for a pair of lifts E and $\mathfrak{H}$ such that

$$\mathrm{ad}_E^k G \equiv 0 \quad \mathrm{mod}\ \widehat{K}_{k-1} \tag{3.14}$$

For this as before take some pair of lifts E and $\mathfrak{H}$ and assume that they satisfy (3.13) with some function υ. Take other lifts $\widetilde{E}$ and $\widetilde{\mathfrak{H}}$. Then relation (3.5) holds for some functions α, β,γ, δ. By direct computations, using relations (3.2) and (3.10), one gets

$$\mathrm{ad}^k_{\widetilde{E}}G \equiv \Big(\upsilon + k\big((\frac{k-1}{2}+w_D)\alpha + 2\beta\big)\Big)\mathrm{ad}^{k-1}_{\widetilde{E}}G \mod \widehat{K}_{k-1} \tag{3.15}$$

Thus, a pair of lifts $\widetilde{E}$ and $\widetilde{\mathfrak{H}}$ satisfies condition (3.14) if and only if

$$(\frac{k-1}{2}+w_D)\alpha + 2\beta = -\frac{\upsilon}{k}. \tag{3.16}$$

We see that linear equations (3.7) and (3.16) (w.r.t. α and β) are linearly independent if and only if $w_D \neq \frac{k+1}{2}$. Hence in the case $i_D = 1$ and $w_D \neq \frac{k+1}{2}$ conditions (3.4) and (3.14) fix uniquely the lift of V to the bundle B_1.

It remains to consider the case $i_D = 1$ and $w_D = \frac{k+1}{2}$. In this case in general V cannot be lifted to B_1 canonically, but one can find the canonical lift of V to B_2. First define the canonical lift of V to the bundle B. Take $\mu = \big(\lambda, [x_1 : x_2]\big) \in B$ where $\lambda = (p, q) \in \mathbb{P}(D^\perp)$, $[x_1 : x_2]$ are homogeneous coordinates on $\mathbb{P}(D^\perp(q))$ such that $[p] = [x_1 : 0]$. Then $\varphi([p]) = \frac{x_2}{x_1}$ defines coordinates on $\mathbb{P}(D^\perp(q))$. Consider the curve

$$\Upsilon_\mu(t) = \big((\varphi^{-1}(t), q), [x_1 : (x_2 - tx_1)]\big). \tag{3.17}$$

Then the canonical lift E of V to B is defined by

$$E(\mu) = \frac{d}{dt}\Upsilon_\mu(t)|_{t=0}. \tag{3.18}$$

Now we are ready to define the canonical lift of V to B_2. For this let as before $\mathfrak{P} : B_2 \to B$ be the canonical projection and consider all lifts E of V to B_2 such that $\mathfrak{P}_*(E)$ is the canonical lift of V to B. If E and $\tilde{E}$ are two such lifts then they are related as in (3.12) for some function b. By analogy with above the normalization condition (3.7) transforms to the condition $\beta = -\frac{\eta}{k+1}$ and gives the canonical lift of V to B_2. By this we have completed to lift V to the corresponding bundles B_i in all possible cases.

Note that by direct computation one has that the canonical lift E to B_2 of V satisfies the following relations:

$$[\mathbf{a}, E] = E, \quad [\mathbf{b}, E] = 0, \quad [\mathbf{c}, E] = -2\mathbf{a}. \tag{3.19}$$

Note also that the first two relations are valid for the canonical lift of V to B_1 as well. For this, using (3.2), it is enough to show that the line distribution generated by the canonical lift E is invariant with respect to the flows generated by the vector fields $\mathbf{a}$ and $\mathbf{b}$. The latter follows from the normalization conditions (3.4) and (3.14) and the fact that the distribution $\widehat{L}_{k-1}$ is invariant w.r.t. to these flows.

Relations (3.1) and (3.19) imply that the vector fields $\mathbf{a}, \mathbf{c}, E$ form the Lie algebra isomorphic to $\mathfrak{sl}_2(\mathbb{R})$, and together with $\mathbf{b}$ they form the Lie algebra isomorphic to $\mathfrak{gl}_2(\mathbb{R})$.

Step 2. The canonical lift of C. We assume that E is the canonical lift of V to the corresponding bundle B_i defined in Step 1 and $\mathfrak{H}$ is a lift of C to the same B_i. As before, let G be a section of K_1 satisfying (3.8). Define

$$F = [E, [\mathfrak{H}, \mathrm{ad}^{k-1}_E G]].$$

Then F is a vector field not contained in $\hat{\Lambda}$. Indeed $\mathrm{ad}^{k-1}_E G$ is out of $\hat{K}$ and thus $[\mathfrak{H}, \mathrm{ad}^{k-1}_E G]$ is out of $\hat{H}$, but in $\hat{\Lambda}$. Then $[E, [\mathfrak{H}, \mathrm{ad}^{k-1}_E G]]$ is out of $\hat{\Lambda}$ since $[V, \Lambda] = T\mathbb{P}(D^\perp)$. There exists a function ξ_0 such that

$$\mathrm{ad}_{\mathfrak{H}} F \equiv \xi_0 F \mod \hat{\Lambda}. \tag{3.20}$$

We are looking for a lift $\mathfrak{H}$ of C (to one of the bundles B_i) satisfying:

$$\mathrm{ad}_{\mathfrak{H}} F \equiv 0 \mod \hat{\Lambda}. \tag{3.21}$$

For this start with some lift $\mathfrak{H}$ to B_1 or B_2 and assume that it satisfy (3.20) for some function ξ_0. Take another lift $\widetilde{\mathfrak{H}}$ of V. Then in the case of a lifting to B_1 there exist functions γ and δ such that

$$\widetilde{\mathfrak{H}} = \mathfrak{H} + \gamma \mathbf{a} + \delta \mathbf{b}, \tag{3.22}$$

while in the case of a lifting to B_2 there is an additional function ρ such that

$$\widetilde{\mathfrak{H}} = \mathfrak{H} + \gamma \mathbf{a} + \delta \mathbf{b} + \rho \mathbf{c}. \tag{3.23}$$

Then in both cases by direct computations, using relations (3.2), we get

$$\mathrm{ad}_{\tilde{\mathfrak{H}}} \tilde{F} \equiv \mathrm{ad}_{\mathfrak{H}} F + ((k + w_D - i_D + 1)\gamma + 3\delta)\, f \mod \hat{\Lambda},$$

where $\tilde{F} = [E, [\tilde{\mathfrak{H}}, \mathrm{ad}_E^{k-1} \tilde{G}]]$. Thus, the lift $\widetilde{\mathfrak{H}}$ satisfies condition (3.21) if and only if

$$(k + w_D - i_D + 1)\gamma + 3\delta = -\xi_0. \tag{3.24}$$

If $i_D > 1$ then we have proved in Step 1 that the bundle B_1 is reduced to B_3 and then $\mathfrak{H}$ is defined uniquely modulo $\mathbf{b}$. Therefore, if $i_D > 1$ then equation (3.24) is reduced to $\delta = -\frac{\xi_0}{3}$, which determines the canonical lift of C.

If $i_D = 1$ then we are looking for one more normalization condition in addition to (3.21) in the case $w_D \neq \frac{k+1}{2}$ and two more normalization conditions in the case $w_D = \frac{k+1}{2}$. Let us assume first $w_D = 1$. Then $w_D \neq \frac{k+1}{2}$ since $k > 1$. Moreover, we can take $G = [\mathfrak{H}, [E, \mathfrak{H}]]$. Then, since $k > 1$, $[\mathfrak{H}, G] \in \Gamma(\hat{H})$. The distribution $\hat{H}$ modulo $\hat{H}_{k+1}$ is spanned by $G, \mathrm{ad}_E G, \ldots, \mathrm{ad}_E^{k-1} G$. If we consider another lift $\tilde{\mathfrak{H}}$ and the corresponding $\hat{G}$ then $\mathrm{ad}_E^i \tilde{G} = \mathrm{ad}_E^i G \mod \hat{H}_{k+1}$ for any i. Therefore the sub-distribution

$$\mathfrak{M} = \mathrm{span}\{\mathrm{ad}_E^i G \mid i = 1, \ldots, k-1\} + \hat{H}_{k+1} \subset \hat{H} \tag{3.25}$$

is well defined. We stress that G is not a section of $\mathfrak{M}$. Since $G \in \Gamma(\hat{K})$, there exists a function ξ_1 such that

$$\mathrm{ad}_{\mathfrak{H}} G \equiv \xi_1 G \mod \mathfrak{M}. \tag{3.26}$$

Our additional normalization condition for a lift $\mathfrak{H}$ is

$$\mathrm{ad}_{\mathfrak{H}} G \equiv 0 \mod \mathfrak{M}. \tag{3.27}$$

Clearly $\mathrm{ad}_{\mathfrak{H}} G = -\mathrm{ad}_{\mathfrak{H}}^3 E$. If we take another lift $\widetilde{\mathfrak{H}}$, then it satisfies (3.22) or (3.23) for some functions γ, δ, and ρ. By direct computations we get

$$\mathrm{ad}_{\tilde{\mathfrak{H}}}^3 E \equiv \mathrm{ad}_{\mathfrak{H}}^3 E - 3\,(\gamma + \delta)\, G \mod \widehat{H}_{k+1}.$$

Therefore the lift $\widetilde{\mathfrak{H}}$ satisfies condition (3.27) if and only if

$$\gamma + \delta = \frac{\xi_1}{3}. \tag{3.28}$$

Equations (3.24) and (3.28) are independent if and only if $k \neq 2$ (recall that we assume here that $w_D = i_D = 1$). However, Corollary 2.1 says that if $k = 2$ then $i_D > 1$. In this way conditions (3.21) and (3.27) determine the canonical lift of C to B_1 in the case $w_D = i_D = 1$.

If $i_D = 1$ and $w_D > 1$ then $[\mathfrak{H}, [\mathfrak{H}, E]] \in \hat{H}_{k+1}$. By Lemma 2.1 $w_D \geq 3$. Similarly to the previous case of $i_D = 1$ we have a sub-distribution

$$\mathfrak{N} = \mathrm{span}\{\mathrm{ad}_E^i \mathfrak{H} \mid i = 1, \ldots, k\} + \widehat{V} \subset \hat{H}_{k+1}.$$

There exists a function ξ_2 such that

$$\mathrm{ad}_{\mathfrak{H}}^2 E \equiv \xi_2 \mathrm{ad}_{\mathfrak{H}} E \mod \mathfrak{N}. \tag{3.29}$$

Our additional normalization condition for a lift $\mathfrak{H}$ in this case is

$$\mathrm{ad}^2_{\mathfrak{H}}E \equiv 0 \mod \mathfrak{N}. \tag{3.30}$$

If we take another lift $\widetilde{\mathfrak{H}}$ then it satisfies (3.22) or (3.23) for some functions γ, δ, and ρ. By direct computations, using relations (3.2), we get

$$\mathrm{ad}^2_{\widetilde{\mathfrak{H}}}E \equiv \mathrm{ad}^2_{\mathfrak{H}}E + 2\left(\gamma + \frac{1}{2}\delta\right)\mathrm{ad}_{\mathfrak{H}}E \mod \widehat{V}.$$

Therefore the lift $\widetilde{\mathfrak{H}}$ satisfies condition (3.30) if and only if

$$\gamma + \frac{1}{2}\delta = -\frac{\xi_2}{2}. \tag{3.31}$$

Equations (3.24) and (3.31) are independent if and only if $k + w_D \neq 6$. On the other hand, if $k + w_D = 6$ and $w_D > 1$ then $k = w_D = 3$. However, this situation cannot occur if $[D_{k+1}, D_{k+1}] = D$. Indeed, assume that $k = w_D = 3$. Let ε be a section of V and h be a section of C. Then

$$[h, \mathrm{ad}_\varepsilon h] = 0 \mod H_4, \quad [h, \mathrm{ad}^2_\varepsilon h] = 0 \mod H_4. \tag{3.32}$$

Applying ad_ε to the last relation, we get that

$$[\mathrm{ad}_\varepsilon h, \mathrm{ad}^2_\varepsilon h] + [h, \mathrm{ad}^3_\varepsilon h] \in H_4. \tag{3.33}$$

Applying ad_ε to (3.33) and using the fact that $\mathrm{ad}^4_\varepsilon h \in H_4 = \mathrm{span}\{h, \varepsilon, \mathrm{ad}_\varepsilon h, \mathrm{ad}^2_\varepsilon h, \mathrm{ad}^3_\varepsilon h\}$ and relations (3.32), we get

$$[\mathrm{ad}_\varepsilon h, \mathrm{ad}^3_\varepsilon h] \in \mathbb{R}[h, \mathrm{ad}^3_\varepsilon h] + H_4. \tag{3.34}$$

Finally applying ad_ε to the last relation and using (3.33) we obtain that

$$[\mathrm{ad}^2_\varepsilon h, \mathrm{ad}^3_\varepsilon h] \in \mathbb{R}[h, \mathrm{ad}^3_\varepsilon h] + H_4.$$

Thus $\dim[H_4, H_4]/H_4 = 1$ and $[H_4, H_4] \neq H$, which implies that $[D_4, D_4] \neq D$ in contradiction to our genericity assumption (G2). So, the case $k = w_D = 3$ is impossible.

As a conclusion, in the case when $i_D = 1$, $w_D > 1$, and $w_D \neq \frac{k+1}{2}$ conditions (3.21) and (3.30) determine the canonical lift of C to the bundle B_1, while in the case when $i_D = 1$ and $w_D = \frac{k+1}{2}$ the same conditions determine a lift of C to the bundle B_2 modulo $\mathbb{R}\mathbf{c}$. It remains to kill the freedom in the latter case by introducing one more normalization condition. For this take a lift $\mathfrak{H}$ of C to B_2 satisfying conditions (3.21) and (3.30). One can take $G = [\mathfrak{H}, (ad_E)^{\frac{k+1}{2}}\mathfrak{H}]$. Since $k \equiv 1 \mod 4$ and $k > 1$, then $k \geq 5$ and therefore $[\mathfrak{H}, [E, G]]$ is a section of H. Hence there exists a function ξ_3 such that

$$[\mathfrak{H}, [E, G]] \equiv \xi_3 G \mod \mathfrak{M}, \tag{3.35}$$

where $\mathfrak{M}$ is as (3.25) Our last normalization condition for a lift $\mathfrak{H}$ in the considered case is

$$[\mathfrak{H}, [E, G]] \equiv 0 \mod \mathfrak{M}. \tag{3.36}$$

If we take another lift $\widetilde{\mathfrak{H}}$ satisfying satisfying conditions (3.21) and (3.30), then there exists a function ρ such that

$$\widetilde{\mathfrak{H}} = \mathfrak{H} + \rho\mathbf{c} \tag{3.37}$$

Let $\widetilde{G} = [\widetilde{\mathfrak{H}}, (ad_E)^{\frac{k+1}{2}}\widetilde{\mathfrak{H}}]$. Then by direct computations, using relations (3.19), we get

$$[\widetilde{\mathfrak{H}}, [E, \widetilde{G}]] \equiv [\mathfrak{H}, [E, G]] - (k+1)\rho G \widehat{H}_{k+1}.$$

Therefore the lift $\widetilde{\mathfrak{H}}$ satisfies condition (3.36) if and only if $\rho = \frac{\xi_3}{k+1}$. Hence, conditions (3.21),(3.30), and (3.36) fix the lift of C to the bundle B_2 uniquely. By this we have completed to lift C to the corresponding bundles B_i in all possible cases.

Finally it is not hard to show that the canonical lift $\mathfrak{H}$ (either to B_1 or to B_2) satisfies the following commutative relations:

$$[\mathbf{a},\mathfrak{H}]=0,\quad [\mathbf{b},\mathfrak{H}]=\mathfrak{H},\quad [\mathbf{c},\mathfrak{H}]=0. \tag{3.38}$$

To prove these relations one can use arguments similar to those used at the end of step 1 for relations (3.19): the distributions $\widehat{\Lambda}$, $\mathfrak{M}$, and $\mathfrak{N}$, appearing in the normalization conditions (3.21), (3.27), (3.30), and (3.36), are invariant with respect to the flow generated by vector fields $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$.

Step 3. Construction of the canonical frame. Now let E and $\mathfrak{H}$ be the canonical lift constructed in the previous steps. We can complete E, $\mathfrak{H}$ and the tuple consisting of the fundamental vertical vector fields of the corresponding bundle B_i to the canonical frame B_i by taking appropriate iterative Lie brackets of E and $\mathfrak{H}$.

More precisely, if $i_D\equiv 1$ and w_D is constant and not equal to $\frac{k+1}{2}$ as a canonical frame associated with our distribution on the bundle B_1 we can take the tuple of vector fields

$$\Big(E,\mathfrak{H},\mathrm{ad}_E\mathfrak{H},\dots,\mathrm{ad}_E^k\mathfrak{H},G,\dots,\mathrm{ad}_E^{k-1}G,[\mathfrak{H},\mathrm{ad}_E^{k-1}G],\big[E,[\mathfrak{H},\mathrm{ad}_E^{k-1}G]\big],\mathbf{a},\mathbf{b}\Big), \tag{3.39}$$

where $G=[\mathfrak{H},\mathrm{ad}_E^{w_D}\mathfrak{H}]$. If $i_D\equiv 1$ and $w_D\equiv\frac{k+1}{2}$ then as a canonical frame associated with our distribution on the bundle B_2 we can take the tuple of the vectors

$$\Big(E,\mathfrak{H},\mathrm{ad}_E\mathfrak{H},\dots,\mathrm{ad}_E^k\mathfrak{H},G,\dots,\mathrm{ad}_E^{k-1}G,[\mathfrak{H},\mathrm{ad}_E^{k-1}G],\big[E,[\mathfrak{H},\mathrm{ad}_E^{k-1}G]\big],\mathbf{a},\mathbf{b},\mathbf{c}\Big), \tag{3.40}$$

Further, if $i_D>1$, since $H_{k+1}^2=H$, we can complete the tuple $(E,\mathfrak{H},\mathrm{ad}_E\mathfrak{H},\dots,\mathrm{ad}_E^k\mathfrak{H},b)$ to the canonical frame on B_3 by k vector fields of the type $[\mathrm{ad}_E^s\mathfrak{H},\mathrm{ad}_E^t\mathfrak{H}]$ for some integer s, t, a vector fields of the type $\big[\mathfrak{H},[\mathrm{ad}_E^{\bar s}\mathfrak{H},\mathrm{ad}_E^{\bar t}\mathfrak{H}\big]$ and a vector field of the type $\Big[E,\big[\mathfrak{H},[\mathrm{ad}_E^{\bar s}\mathfrak{H},\mathrm{ad}_E^{\bar t}\mathfrak{H}]\big]\Big]$ for some integers $\bar s$ and $\bar t$. By this we have completed the construction of the canonical frame for corank 2 distributions of the considered in all 3 cases.

Finally, since the fundamental vector fields and the vector field E constitute the frame on each fiber of the bundle $\pi\circ\Pi_i:B_i\mapsto M$ and these vector fields are the part of the canonical frame, a diffeomorpism of B_i, sending the canonical frame of a corank 2 distribution D to the canonical frame of a corank 2 distribution D' (by the pushforward) is fiberwise. Therefore it induces the diffeomorphism of M. The latter diffeomorphism induces the equivalence between the distributions D and D', because $(\pi\circ\Pi_i)_*\mathrm{span}\{\mathfrak{H},\mathrm{ad}_E\mathfrak{H},\dots,\mathrm{ad}_E^k\mathfrak{H}\}=D_{k+1}$ and $D_{k+1}^2=D$. The proof of Theorem 3.1 is completed. □

4. Symmetric models

In this section given $k>2$ we find all maximally symmetric models for $(2k+1,2k+3)$-distributions satisfying conditions (G1) and (G2) with respect to the local equivalence. We show that the algebra of infinitesimal symmetries for this models is $(2k+6)$- dimensional if $k\neq 1 \mod 4$ and $(2k+7)$-dimensional if $k\equiv 1 \mod 4$, i.e. the upper bounds of Corollary 3.1 are sharp. By Theorem 2 it may occur only if $i_D\equiv 1$. Note that the case $k=2$ is exceptional, because by Corollary 2.1 in this case i_D has to be equal to 2. As was already mentioned in the Introduction, the most symmetric model for $k=2$ (given by (1.7)) can be obtained from the analysis of our canonical frame on B_3 described in the proof of Theorem 3.1 but this model can be also recognized without difficulties from the list of 7-dimensional non-degenerate fundamental graded Lie algebra given in [10], thus we omit this analysis.

So, let $k>2$, $i_D\equiv 1$, and $w_D\equiv w$. Then the canonical frame is given by the tuple of vector fields (3.39) if $w\neq\frac{k+1}{2}$ and by the tuple of vector fields (3.40) if $w=\frac{k+1}{2}$. For shortness let

$$\begin{aligned}
&\mathbf{x}_j = \operatorname{ad}_E^j \mathfrak{H}, \quad 0 \le j \le k,\\
&\mathbf{y}_j = \operatorname{ad}_E^{j-1} G, \quad 1 \le j \le k,\\
&\mathbf{z} = [\mathfrak{H}, \operatorname{ad}_E^{k-1} G], \quad \mathbf{n} = \big[E, [\mathfrak{H}, \operatorname{ad}_E^{k-1} G]\big].
\end{aligned}$$

Then in the new notation

$$\begin{aligned}
&[E, \mathbf{x}_j] = \mathbf{x}_{j+1}, \quad 0 \le j \le k-1,\\
&[\mathbf{x}_0, \mathbf{x}_w] = \mathbf{y}_1, \quad [E, \mathbf{y}_j] = \mathbf{y}_{j+1},\ 1 \le j \le k-1,\\
&[\mathbf{x}_0, \mathbf{y}_k] = \mathbf{z}, \quad [E, \mathbf{z}] = \mathbf{n}.
\end{aligned} \tag{4.1}$$

Denote by $\mathrm{MS}(k, w)$ the set of all equivalence classes of germs of $(2k+1, 2k+3)$-distributions D, satisfying conditions (G1) and (G2), relations $i_D \equiv 1$ and $w_D \equiv w$, and having the algebra of infinitesimal symmetries of the dimension equal to the dimension of the bundle, where their canonical frames are constructed. Take a distribution D representing an element of $\mathrm{MS}(k, w)$. This happens if and only if *all structural functions of the canonical frame of D are constant.* In other words, the vector fields of the canonical frame of D should form the Lie algebra over $\mathbb{R}$ (that is isomorphic to the algebra of infinitesimal symmetries of the distribution D). Denote this algebra by $\mathfrak{g}$. What properties does this algebra have? First, combining (4.1) with (3.19) and (3.38) (with $\mathfrak{H}$ replaced by $\mathbf{x}_0$) and using the Jacobi identity, one gets

(4.2)

$$\begin{aligned}
&[\mathbf{a}, \mathbf{x}_j] = j\mathbf{x}_j, \quad [\mathbf{a}, \mathbf{y}_j] = (w+j-1)\mathbf{y}_j, \quad [\mathbf{a}, \mathbf{z}] = (w+k-1)\mathbf{z}, \quad [\mathbf{a}, \mathbf{n}] = (w+k)\mathbf{n},\\
&[\mathbf{b}, \mathbf{x}_j] = \mathbf{x}_j, \quad [\mathbf{b}, \quad \mathbf{y}_j] = 2\mathbf{y}_j, \quad [\mathbf{b}, \mathbf{z}] = 3\mathbf{z}, \quad [\mathbf{b}, \mathbf{n}] = 3\mathbf{n}.
\end{aligned}$$

This motivates the introduction of the following natural *bi-grading* on the algebra $\mathfrak{g}$ by assigning to each element of the tuple (3.39) or (3.40) two integer numbers as follows:

$$\begin{aligned}
&\mathbf{x}_j \mapsto (-j, -1),\ \mathbf{y}_j \mapsto (-(w+j-1), -2),\\
&\mathbf{z} \mapsto (-(w+k-1), -3),\ \mathbf{n} \mapsto (-w-k, -3)
\end{aligned} \tag{4.3}$$

$$E \mapsto (-1, 0), \quad \{\mathbf{a}, \mathbf{b}\} \mapsto (0, 0), \quad \mathbf{c} \mapsto (1, 0). \tag{4.4}$$

The above assignment for elements in (4.3) is given by the following simple rule: the first integer in the bi-degrees given by (4.3) is the number of appearance of E in the representation of the corresponding vector field from the canonical frame as the iterative brackets of E multiplied by -1 and $\mathfrak{H}$ and the second integer there is the number of appearance of E in this representation multiplied by -1. Let $\mathfrak{g}_{j_1, j_2}$ be the linear span (over $\mathbb{R}$) of all elements of the canonical frame corresponding to the pair (j_1, j_2). Then using relations (3.19), (3.38) and the Jacobi identity, one gets that

$$[\mathfrak{g}_{j_1, j_2}, \mathfrak{g}_{l_1, l_2}] \subset \mathfrak{g}_{j_1+l_1, j_2+l_2},$$

i.e $\mathfrak{g} = \bigoplus_{(j_1, j_2) \in \mathbb{Z}^2} \mathfrak{g}_{j_1, j_2}$ is indeed the bi-grading of the Lie algebra $\mathfrak{g}$.

Definition 4.1. *Given $k > 2$ and odd w , $1 \le w \le 2k-1$, a bi-graded Lie algebra $\mathfrak{g}$ is called a bi-graded Lie algebra of the type (k, w) if the following two conditions hold*

(1)

$$\mathfrak{g} = \begin{cases} \operatorname{span}\{E, \mathbf{x}_0, \dots, \mathbf{x}_k, \mathbf{y}_1, \dots, \mathbf{y}_k, \mathbf{z}, \mathbf{n}, \mathbf{a}, \mathbf{b}\} & \text{if } w \ne \frac{k+1}{2},\\ \operatorname{span}\{E, \mathbf{x}_0, \dots, \mathbf{x}_k, \mathbf{y}_1, \dots, \mathbf{y}_k, \mathbf{z}, \mathbf{n}, \mathbf{a}, \mathbf{b}, \mathbf{c}\} & \text{if } w = \frac{k+1}{2} \end{cases}. \tag{4.5}$$

such that the commutative relations (4.1), (3.1), (3.19), *and* (3.38) *(with $\mathfrak{H}$ replaced by $\mathbf{x}_0$ in the latter) hold;*

(2) *the bi-grading on $\mathfrak{g}$ is given by* (4.3)-(4.4).

So we have shown that *if the distribution D representing an element of* $\mathrm{MS}(k,w)$ *then its algebra of infinitesimal symmetries* $\mathrm{symm}(D)$ *is a bi-graded Lie algebra of type* (k,w).

Now let

$$\mathfrak{m} = \bigoplus_{j_2<0} \mathfrak{g}_{j_1,j_2} = \mathrm{span}\{\mathbf{x}_0,\dots,\mathbf{x}_k,\mathbf{y}_1,\dots,\mathbf{y}_k,\mathbf{z},\mathbf{n}\}, \tag{4.6}$$

$$\mathfrak{g}' = \bigoplus_{j_2\geq 0} \mathfrak{g}_{j_1,j_2}.$$

Note that $\mathfrak{g}' = \mathrm{span}\{E,\mathbf{a},\mathbf{b}\}$ if $w_D \neq \frac{k+1}{2}$ and $\mathfrak{g}' = \mathrm{span}\{E,\mathbf{a},\mathbf{b},\mathbf{c}\}$ if $w_D = \frac{k+1}{2}$. Also note that both $\mathfrak{m}$ is a bi-graded nilpotent subalgebra of $\mathfrak{g}$. Besides,

$$\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{m}. \tag{4.7}$$

By the standard arguments the distribution D is locally equivalent to an invariant distribution on the homogeneous space G/G', where G and G' are the connected, simply-connected Lie groups with the Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$, respectively. Moreover, from the splitting (4.7) it follows that *the distribution D is locally equivalent to the left invariant distribution $D_{\mathfrak{g}}$ on the connected, simply connected Lie group $\mathcal{M}$ with the Lie algebra m such that*

$$D_{\mathfrak{g}}(e) = \mathrm{span}\{\mathbf{x}_0,\dots,\mathbf{x}_k,\mathbf{y}_1,\dots,\mathbf{y}_k\}, \tag{4.8}$$

where e is the identity of the group $\mathcal{M}$. Moreover, we have the following

Proposition 4.1. *The correspondence between the set* $\mathrm{MS}(k,w)$ *and the set of all bi-graded Lie algebras of type* (k,w)*, given by* $D \mapsto \mathrm{symm}(D)$*, is a bijection.*

Proof. First we prove the following lemma, which will be also useful for other purposes in the sequel:

Lemma 4.1. (A) *If $\mathfrak{g}$ is a bi-graded Lie algebra of type (k,w) with the basis as in Definition 4.1, then*

$$\begin{aligned} &[\mathbf{x}_i,\mathbf{y}_{k-i}] = (-1)^i\mathbf{z}, \quad 0\leq i\leq k-1\\ &[\mathbf{x}_i,\mathbf{y}_{k-i+1}] = (-1)^{i+1}i\mathbf{n}, \quad 1\leq i\leq k\\ &[\mathbf{x}_i,\mathbf{x}_j] = c_{i,j}\mathbf{y}_{i+j-w+1}, \end{aligned} \tag{4.9}$$

where $c_{i,j}$ satisfy the following four properties (in addition to the evident antisymmetricity $c_{i,j} = -c_{j,i}$):

(1) $c_{i,j} = 0$ *if* $i+j<w$ *or* $i+j>k+w-1$;

(2) $c_{0,w} = 1$;

(3)

$$c_{i,j} = c_{i+1,j} + c_{i,j+1}; \tag{4.10}$$

(4)

$$\begin{aligned} &(-1)^i c_{0,k+w-1-i} - (-1)^{k+w-1-i}c_{0,i} = c_{i,k+w-1-i}\\ &(-1)^{i+1} i c_{0,k+w-i} - (-1)^{k+w-i}(k+w-i+1)c_{0,i} = 0. \end{aligned} \tag{4.11}$$

(B) *Conversely, if $w\neq\frac{k+1}{2}$ and the tuple $(E,\mathbf{x}_0,\dots,\mathbf{x}_k,\mathbf{y}_1,\dots,\mathbf{y}_k,\mathbf{z},\mathbf{n},\mathbf{a},\mathbf{b})$ satisfy relations* (4.1)*,* (3.1)*,* (3.19)*,* (3.38)*, and* (4.9) *with antisymmetric matrix $(c_{i,j})$ satisfy* (4.10) *and* (4.11)*, then this tuple spans the bi-graded Lie algebra of type (k,w).*

(C) *The matrix $(c_{i,j})$ defines the bi-graded Lie algebra of type (k,w) uniquely, up to an isomorphism.*

Proof. Throughout this proof we use the fact that by (4.3) and (4.4) the spaces $\mathfrak{g}_{j_1,j_2}$ are at most one-dimensional if $(j_1, j_2) \neq (0,0)$. Therefore using the compatibility of the Lie brackets with the bi-grading we get that there exists constant a_i, b_i and $c_{i,j}$ such that

$$[\mathbf{x}_i, \mathbf{y}_{k-i}] = b_i\mathbf{z}, \quad 0 \leq i \leq k-1$$
$$[\mathbf{x}_i, \mathbf{y}_{k-i+1}] = a_i\mathbf{n}, \quad 1 \leq i \leq k$$
$$[\mathbf{x}_i, \mathbf{x}_j] = c_{i,j}\mathbf{y}_{i+j-w+1}.$$

Constants a_i and b_i can be found by applying the Jacobi identity to $[E, [\mathbf{x}_i, \mathbf{y}_{k-i}]]$. On the one hand, we get $[E, [\mathbf{x}_i, \mathbf{y}_{k-i}]] = b_i\mathbf{n}$ and on the other hand

$$[E, [\mathbf{x}_i, \mathbf{y}_{k-i}]] = [\mathbf{x}_{i+1}, \mathbf{y}_{k-i}] + [\mathbf{x}_i, \mathbf{y}_{k-i+1}] = (a_i + a_{i+1})\mathbf{n}.$$

Hence we get the equation $a_i + a_{i+1} = b_i$ which holds for any $i = 1, \dots, k-1$. Moreover, if $i = 0$ we get $a_1 = b_0$. Similarly we consider $[E, [\mathbf{x}_i, \mathbf{y}_{k-i-1}]]$ and by Jacobi identity we get the equation $b_i + b_{i+1} = 0$, which holds for any $i = 0, \dots, k-2$. By definition $b_0 = 1$. In this way we get $b_i = (-1)^i$ and then $a_i = i(-1)^{i+1}$.

In order to get the relation $c_{i,j} = c_{i+1,j} + c_{i,j+1}$ we consider Jacobi identity applied to $[E, [\mathbf{x}_i, \mathbf{x}_j]]$, whereas in order to get relations (4.11) we consider Jacobi identity applied to $[\mathbf{x}_0, [\mathbf{x}_i, \mathbf{x}_{k+w-1-i}]]$ and $[\mathbf{x}_0, [\mathbf{x}_i, \mathbf{x}_{k+w-i}]]$. In this way the part (A) of the Lemma is proved.

Let us prove now the part (B). From the part (A) we know that (4.10) and (4.11) are satisfied for any bi-graded Lie algebra of type (k, w). We have to show that there is no other relation on structural constants a_i, b_i and $c_{i,j}$. For $w \neq \frac{k+1}{2}$, an additional possibly non-trivial relation can be obtained from Jacobi identity applied to $[\mathbf{x}_l, [\mathbf{x}_i, \mathbf{x}_j]]$, where $l + i + j = k + w - 1$ or $l + i + j = k + w$. We will show that all these relations are consequences of (4.10) and (4.11).

We can assume that $l \leq i$ and $l \leq j$. The proof is by induction: we assume that Jacobi identity is satisfied for $[\mathbf{x}_{l-1}, [\mathbf{x}_{\tilde{i}}, \mathbf{x}_{\tilde{j}}]]$ where $l - 1 \leq \tilde{i}$ and $l - 1 \leq \tilde{j}$. The case $l = 0$ corresponds to (4.11). If $l > 0$ then $\mathbf{x}_l = [E, \mathbf{x}_{l-1}]$. Thus

$$\begin{aligned}
[\mathbf{x}_l, [\mathbf{x}_i, \mathbf{x}_j]] &= [E, [\mathbf{x}_{l-1}, [\mathbf{x}_i, \mathbf{x}_j]]] - [\mathbf{x}_{l-1}, [E, [\mathbf{x}_i, \mathbf{x}_j]]] \\
&= [E, [\mathbf{x}_{l-1}, [\mathbf{x}_i, \mathbf{x}_j]]] - [\mathbf{x}_{l-1}, [\mathbf{x}_{i+1}, \mathbf{x}_j]] - [\mathbf{x}_{l-1}, [\mathbf{x}_i, \mathbf{x}_{j+1}]].
\end{aligned}$$

We used here (4.10), which is equivalent to Jacobi identity of brackets involving E. By our assumption, we know that Jacobi identity is satisfied by $[\mathbf{x}_{l-1}, [\mathbf{x}_i, \mathbf{x}_j]]$, $[\mathbf{x}_{l-1}, [\mathbf{x}_{i+1}, \mathbf{x}_j]]$ and $[\mathbf{x}_{l-1}, [\mathbf{x}_i, \mathbf{x}_{j+1}]]$. Therefore

$$\begin{aligned}
[\mathbf{x}_l, [\mathbf{x}_i, \mathbf{x}_j]] = \; & [E, [[\mathbf{x}_{l-1}, \mathbf{x}_i], \mathbf{x}_j]] + [E, [\mathbf{x}_i, [\mathbf{x}_{l-1}, \mathbf{x}_j]]] \\
& -[[\mathbf{x}_{l-1}, \mathbf{x}_{i+1}], \mathbf{x}_j] - [\mathbf{x}_{i+1}, [\mathbf{x}_{l-1}, \mathbf{x}_j]] - [[\mathbf{x}_{l-1}, \mathbf{x}_i], \mathbf{x}_{j+1}] - [\mathbf{x}_i, [\mathbf{x}_{l-1}, \mathbf{x}_{j+1}]].
\end{aligned}$$

But, if we use (4.10) for Lie bracket involving E we get

$$[E, [[\mathbf{x}_{l-1}, \mathbf{x}_i], \mathbf{x}_j]] - [[\mathbf{x}_{l-1}, \mathbf{x}_{i+1}], \mathbf{x}_j] - [[\mathbf{x}_{l-1}, \mathbf{x}_i], \mathbf{x}_{j+1}] = [[\mathbf{x}_l, \mathbf{x}_i], \mathbf{x}_j]$$

and

$$[E, [\mathbf{x}_i, [\mathbf{x}_{l-1}, \mathbf{x}_j]]] - [\mathbf{x}_{i+1}, [\mathbf{x}_{l-1}, \mathbf{x}_j]] - [\mathbf{x}_i, [\mathbf{x}_{l-1}, \mathbf{x}_{j+1}]] = [\mathbf{x}_i, [\mathbf{x}_l, \mathbf{x}_j]].$$

This completes the proof of part (B).

To prove part (C) let us take another basis of the algebra $\mathfrak{g}$ as in Definition 4.1 and let $\tilde{E}$, $\tilde{\mathbf{x}}_j$ and $\tilde{\mathbf{y}}_j$ be the corresponding elements of this basis. Then $\tilde{E} = \alpha E$ and $\tilde{\mathbf{x}}_0 = \beta\mathbf{x}_0$ for some α and β. This together with (4.1) implies that $\tilde{\mathbf{x}}_j = \alpha^j\beta\mathbf{x}_j$, $\tilde{\mathbf{y}}_j = \alpha^{w+j-1}\beta^2\mathbf{y}_j$. Then using the last relation of (4.9) we get that $\tilde{c}_{i,j} = c_{i,j}$, where $\tilde{c}_{i,j}$ denotes the corresponding constant for the new basis. So, each $c_{i,j}$ is an invariant of the bi-graded Lie algebra of type (k, w), which complete the proof of the last part of the lemma.

□

Now fix a bi-graded Li algebra $\mathfrak{g}$ of type (k,w). Let $\mathfrak{m}$ be as in (4.6), and let $D_{\mathfrak{g}}$ be the left-invariant distribution on the Lie group $\mathcal{M}$ with the Lie algebra $\mathfrak{m}$, defined by relation (4.8). Then by the first two relations of (4.9) it satisfies condition (G1). Further, assume by contradiction that $D_{\mathfrak{g}}$ does not satisfy condition (G2). Then from the last relation of (4.9) it follows that there exists l, $w \leq l \leq k+w-1$ such that $c_{i,j}=0$ for all i,j such that $i+j=l$. But from relation (4.10) it follows that $c_{i,j}=0$ for all i,j such that $i+j\leq l$ in contradiction with condition (2) from Lemma 4.1. Conditions (1) and (2) from Lemma 4.1 also imply that $w_{D_{\mathfrak{g}}}=w$. It is also clear by constructions that the group G is a subgroup of the group of symmetries of $D_{\mathfrak{g}}$. This together with Corollary 3.1 implies that the algebra $\mathrm{symm}(D_{\mathfrak{g}})$ of infinitesimal symmetries of $D_{\mathfrak{g}}$ is isomorphic to $\mathfrak{g}$ as a Lie algebra. Moreover $\mathrm{symm}(D_{\mathfrak{g}})$ has natural grading ([13], [15]) and the algebras $\mathrm{symm}(D_0)$ and $\mathfrak{g}$ are isomorphic as graded Lie algebras, where the grading on $\mathfrak{g}$ is considered with respect to the second bi-degree. Besides, using Definition 4.1 and relations (4.2) it is not hard to show that $\mathrm{symm}(D_{\mathfrak{g}})$ and $\mathfrak{g}$ are isomorphic as bi-graded Lie algebras. It shows that the correspondence $D\mapsto \mathrm{symm}(D)$ between the set of all bi-graded Lie algebras of type (k,w), given by $D\mapsto \mathrm{symm}(D)$, is a bijection (with the inverse given by $\mathfrak{g}\mapsto D_{\mathfrak{g}}$). This completes the proof of the proposition. $\square$

Now let $\mathfrak{L}_{k,w}$ denote the set of all bi-graded Lie algebras of type (k,w). Let $\mathfrak{L}_k=\bigcup\limits_{w}\mathfrak{L}_{k,w}$, if $k\not\equiv 1 \mod 4$ and $\mathfrak{L}_k=\mathfrak{L}_{k,\frac{k+1}{2}}$ if $k\equiv 1 \mod 4$. From Proposition 4.1 it follows that if the set $\mathfrak{L}_k$ is not empty, then the problem of finding of all maximally symmetric models of $(2k+1,2k+3)$-distributions from the considered class is reduced to the description of the set $\mathfrak{L}_k$. In the sequel we will do a little bit more, describing the sets $\mathfrak{L}_{k,w}$ including the case when $k\equiv 1 \mod 4$ but $w\neq \frac{k+1}{2}$. In particular, the set $\mathfrak{L}_k$ is not empty for any $k>2$ so that Proposition 4.1 gives a way to describe the maximally symmetric models. Set

$$d(k,w)=\begin{cases}\left[\frac{l-w+1}{3}\right] & \text{if } k=2l+1\\ \left[\frac{l-w-1}{3}\right] & \text{if } k=2l\end{cases}. \tag{4.12}$$

The main result of this section is the following

Theorem 4.1. *The set of bi-graded Lie algebras of type (k,w) is $d(k,w)$-parametric family.*

Remark 4.1. In particular, if $d(k,w)=0$, then there exists the unique bi-graded Lie algebra of the type (k,w), while if $d(k,w)<0$, then the set of bi-graded Lie algebras of the type (k,w) is empty. $\square$

The proof of this theorem together with Lemma 4.1 will give a rather explicit description of all these Lie algebras. As a direct consequence of Theorem 4.1 and Proposition 4.1 we have the following

Corollary 4.1. *Let $k>2$*

(1) *If $k\equiv 1 \mod 4$ and $w=\frac{k+1}{2}$ then there is a unique, up to a local equivalence, $(2k+1,2k+3)$-distribution D satisfying conditions (G1) and (G2), which have $(2k+7)$-dimensional infinitesimal symmetry algebra.*

(2) *If w is odd and $w\neq\frac{k+1}{2}$, then the set of $(2k+1,2k+3)$-distributions D from the considered class that satisfy $w_D=w$ and have $(2k+6)$-dimensional infinitesimal symmetry algebra is a $d(k,w)$-parametric family.*

If $k\not\equiv 1 \mod 4$ the families of distributions from the item (2) above are the only distributions from the considered class with $(2k+6)$-dimensional infinitesimal symmetry algebra.

Proof of Theorem 4.1. By Lemma 4.1 in the case $w \neq \frac{k+1}{2}$ the proof of the theorem is reduced to the search of all antisymmetric matrix $(c_{i,j})$ satisfying conditions (1)-(4) of Lemma 4.1, while in the case $w = \frac{k+1}{2}$ Lemma 4.1 guarantees that all bi-graded Lie algebras of type (k,w) are obtained from a subset of such matrices. In a series of lemmas below we bring conditions (3) and (4) of Lemma 4.1 in more convenient form. By relation (4.10), all coefficients $c_{i,j}$ are completely determined by $c_{w-1+i,k-i}$ for $i = 0, \dots, k-w+1$. Denote

$$x_i = \binom{k+w-1}{i+w-1} c_{i+w-1,k-i}. \tag{4.13}$$

Since $(c_{i,j})$ is antisymmetric, we have

$$x_i + x_{k-w+1-i} = 0, \quad i = 0, 1, \dots, \left[\frac{k-w+1}{2}\right]. \tag{4.14}$$

Lemma 4.2. *Systems* (4.10) *and* (4.11) *imply*

$$(-1)^i \sum_{j=0}^{i} x_j = \sum_{j=0}^{i} \binom{k-j}{i-j} x_j, \tag{4.15}$$

for $i = 0, \dots, k-w+1$.

Proof. Denote $y_i = c_{i+w-1,k-i}$. We will use (4.11) and express $c_{0,i}$ in terms of y_i. At the beginning $c_{0,k} = y_0$, as follows from the first equation of (4.11) with $i = w-1$. Then the second equation of (4.11) gives $c_{0,w} = \frac{w}{k}(-1)^{k+1} y_0$. In the next step we again use the first equation of (4.11) with $i = w$ and get $c_{0,k-1} = -y_1 - \frac{w}{k} y_0$. Then we proceed by induction and get the formula:

$$c_{0,k-i} = (-1)^i \sum_{j=0}^{i} \frac{\binom{k+w-1}{j+w-1}}{\binom{k+w-1}{i+w-1}} y_j.$$

On the other hand it follows from (4.10) that

$$c_{0,k-i} = \sum_{j=0}^{i} \binom{i+w-1}{j+w-1} y_j.$$

Now, if we substitute $x_i = \binom{k+w-1}{i+w-1} c_{i+w-1,k-i}$, compare the two expressions for $c_{o,i}$ and use the formula

$$\binom{i+w-1}{j+w-1}\binom{k+w-1}{i+w-1} = \binom{k+w-1}{j+w-1}\binom{k-j}{i-j}$$

we get the desired system (4.15). □

Remark 4.2. From the proof of Lemmas 4.2 it is not hard to see that the space of common solutions of systems (4.14) and (4.15) is in one-to one correspondence with the space of antisymmetric matrices (c_ij), satisfying conditions (1), (3), and (4) of Lemma 4.1 and the correspondence is given by relation (4.13). *Box*

Now we analyze the solution space of system (4.15).

Lemma 4.3. *The solution space of* (4.15) *is isomorphic to the solution space of the system*

$$\sum_{j=0}^{i-1} \binom{w+i}{w+j} x_j = 0, \tag{4.16}$$

for $i = 2, 4, 6, \dots, 2\left[\frac{k-w+2}{2}\right]$. *Moreover, the isomorphism preserves the solution space of* (4.14).

Proof. If we sum equations corresponding to the indices $i-1$ and i from the system (4.15) for $i = 1, \dots, k-w+1$, we get the following system of equations

$$\sum_{j=0}^{i-1} \binom{k+1-j}{i-j} x_j + \gamma_i x_i = 0, \tag{4.17}$$

where $\gamma_i = 0$ if i is even and $\gamma_i = 2$ if i is odd. The first equation from the system (4.15) with $i = 0$ is trivial and we can cross it out. If k is odd we consider the additional equation with $i = k-w+2$

$$\sum_{j=0}^{k-w+1} \binom{k+1-j}{k-w+1-j} x_j = 0. \tag{4.18}$$

We will show later that this equation is a consequence of the other equations from the system (4.17).

For any $l = 0, \dots, \left[\frac{k-w}{2}\right]$ the following tuple

$$\{x_j\}_{j=0}^{2\left[\frac{k-w+2}{2}\right]-1} = \left(\underbrace{0, \dots, 0}_{2l \text{ times}}, \left\{ (-1)^j \binom{k-2l+2}{j-2l+1} \right\}_{j=2l}^{2\left[\frac{k-w+2}{2}\right]-1} \right) \tag{4.19}$$

is the solution of the system (4.17) (note that if k is even, then x_{k-w+1} is not involved in system (4.17)). Indeed, substituting it to this system we get

(4.20)
$$\sum_{j=2l-1}^{i} (-1)^j \binom{k+1-j}{i-j} \binom{k-2l+2}{j-2l+2} = \sum_{j=2l-1}^{i} (-1)^j \frac{(k-2l+2)!}{(k-i+1)!(j-2l+1)!(i-j)!}.$$

But the right-hand side of the last identity is equal to 0. To prove this fact express $t^{k-2l+2} = (t-1+1)^{k-2l+2}$ and expand $(t-1+1)^{k-2l+2}$ into the trinomial expansion. Then the right-hand side of (4.20) is equal to the coefficient of t^{k-i+1} in this expansion multiplied by -1. Therefore it is equal to 0.

It implies that the rank of the system (4.17) (with additional equation (4.18) in the case of odd k) is at most $\left[\frac{k-w+2}{2}\right]$. On the other hand, from the block lower triangular structure of this system it follows that the equations (4.17) with even i (together with equation (4.18) in the case of odd k) are linearly independent. So, the rank of this system is equal to $\left[\frac{k-w+2}{2}\right]$ and all equations of (4.17) with odd i can be dropped.

Finally, the substitution

$$x_j := \binom{k+w+1}{w+j} x_j \tag{4.21}$$

for $j = 0, \dots, k-w+1$ transform system (4.17) to system (4.16). Moreover $\binom{k+w+1}{w+j} = \binom{k+w+1}{w+(k-w+1-j)}$. Hence, the substitution preserves system (4.14). $\square$

Lemma 4.4. *The solution space of* (4.16) *is isomorphic to the solution space of the system*

$$\sum_{j=0}^{2i-1} \binom{\frac{w-1}{2}+i}{\frac{w+1}{2}-i+j} x_j = 0 \tag{4.22}$$

for $i = 1, 2, \dots, \left[\frac{k-w+2}{2}\right]$, *where* $\binom{a}{b} = 0$ *if* $b < 0$. *Moreover, the isomorphism preserves the solution space of* (4.14).

Proof. From (4.19) and the substitution (4.21) it follows that the solution space of (4.16) is spanned by the following tuples

$$\left(\underbrace{0,\dots,0}_{2l\text{ times}},\left\{(-1)^j\binom{w+j}{w+2l-1}\right\}_{j=2l}^{k-w+1}\right) \tag{4.23}$$

with $l=0,\dots,\left[\frac{k-w}{2}\right]$. We claim that (4.22) has the same solution space. For this first prove the following identity

$$\sum_{j\in\mathbb{Z}}(-1)^j\binom{l}{j}\binom{s+j}{m}=\binom{s}{m-l}, \tag{4.24}$$

where $\binom{a}{b}=0$ if $b<0$ or $b>a$. Consider the following polynomial $f(t)=(t+1)^s t^l$. On the one hand, the coefficient of t^m in f is equal to $\binom{s}{m-l}$. On the other hand,

$$f(t)=(t+1)^s\big((t+1)-1\big)^l=\sum_{j\in\mathbb{Z}}(-1)^j\binom{l}{j}(t+1)^{s+j},$$

so that the coefficient of t^m in f is equal to the left-hand side of (4.24). The proof of identity (4.24) is completed.

Now we substitute a vector (4.23) from the solution space of (4.16) to system (4.22) and use identity (4.24) with $l=s=\frac{w-1}{2}+i$, $j\mapsto\frac{w+1}{2}-i+j$ and $m=w+2l-1$. We get

$$\begin{aligned}\sum_{j=2l}^{2i-1}(-1)^j\binom{\frac{w-1}{2}+i}{\frac{w+1}{2}-i+j}\binom{w+j}{w+2l-1} &= \sum_{j\in\mathbb{Z}}(-1)^j\binom{\frac{w-1}{2}+i}{\frac{w+1}{2}-i+j}\binom{w+j}{w+2l-1}\\ &-\binom{\frac{w-1}{2}+i}{\frac{w-1}{2}-i+2l}=0,\end{aligned}$$

which proves the lemma. ☐

Lemma 4.5. *The following identity holds*

$$\sum_{i=0}^{\mu}\binom{\mu}{i}\binom{\omega+i}{y-i}=\sum_{i=0}^{\mu}\binom{\mu}{i}\binom{\omega+i}{2\mu+w-y-i} \tag{4.25}$$

Proof. Consider the function $g(t)=(1+t)^\omega(1+t+\frac{1}{t})^\mu$. On the one hand,

$$g(t)=(1+t)^\omega\big((1+t)+\frac{1}{t}\big)^\mu=\sum_{i=0}^{\mu}\binom{\mu}{i}(1+t)^{\omega+i}\frac{1}{t^{\mu-i}}$$

and the coefficient of $t^{y-\mu}$ in the expansion of $g(t)$ into the powers of t is equal to the left-hand side of (4.25). On the other hand,

$$g(t)=t^\omega\left(1+\frac{1}{t}\right)^\omega\left(\left(1+\frac{1}{t}\right)+t\right)^\mu=\sum_{i=0}^{\mu}\binom{\mu}{i}\left(1+\frac{1}{t}\right)^{\omega+i}t^{\omega+\mu-i}$$

and the coefficient of $t^{y-\mu}$ in the expansion of $g(t)$ into the powers of t is equal to the right-hand side of (4.25). ☐

Proposition 4.2. *The solution space of system* (4.14)-(4.22) *is* $(d(k,w)+1)$*-dimensional, where* $d(k,w)$ *is as in* (4.12)*.*

Proof. First we prove the following

Lemma 4.6. *The solution space of system* (4.14)-(4.22) *is at least* $(d(k,w)+1)$*-dimensional.*

Proof. We treat the cases $k = 2l+1$ and the case $k = 2l$ separately.

1) Case $k = 2l+1$**.** Fix an integer s such that $0 \le s \le d(k,w)$ and let

$$a_s := \frac{w+1+4s}{2}, \quad m_s := \frac{k-w+2}{2} - s - a_s = \frac{k-2w+1-6s}{2}. \tag{4.26}$$

For $i \ge a_s$ multiply the ith equation of system (4.22) by $\binom{m_s}{i-a_s}$ and sum up all the obtained equations. Taking into account (4), we get the following equation

$$\sum_{i=a_s}^{m_s} \binom{m_s+a_s}{i-a_s} \sum_{j=0}^{2i-1} \binom{\frac{w-1}{2}+i}{a_s-2s-i+j} x_j = 0. \tag{4.27}$$

Substituting $i \mapsto i + a_s$ into (4.27) and taking into account (4), we can write (4.27) as follows

$$\sum_{i=0}^{m_s} \binom{m_s}{i} \sum_{j=0}^{2i+2a_s-1} \binom{w+2s+i}{j-2s-i} x_j = 0. \tag{4.28}$$

Identity (4.25) with $\mu = m_s$, $\omega = w+2s$ and $y = j-2s$ implies that the coefficient of x_j and the coefficient of $x_{2m_s+w+6s-j}$ in (4.28) coincide. Note that by (4) one has $2m_s + w + 6s - j = k - w + 1 - j$. So, the coefficient of of x_j and the coefficient of $x_{k-w+1-j}$ in (4.27) coincide. Thus for any s, $0 \le s \le d(k,w)$, the equation of (4.22) with $i = \frac{k-w+2}{2} - s$ is a linear combination of other equations from the system (4.14)-(4.22), which implies the statement of the lemma for odd k.

2) The case of $k = 2l$**.** This case can be treated similarly. For this fix again an integer s such that $0 \le s \le d(k,w)$ and let

$$a_s := \frac{w+3+4s}{2}, \quad m_s := \frac{k-w+1}{2} - s - a_s = \frac{k-2w-2-6s}{2}.$$

then as before for $i \ge a_s$ multiply the ith equation of system (4.22) by $\binom{m_s}{i-l_s}$ and sum up all the obtained equations and use identity (4.25) (with $\mu = m_s$, $\omega = w+1+2s$, and $y = j-2s-1$) to get that the coefficient of x_j and the coefficient of $x_{k-w+1-j}$ in the considered linear combination of equations from system (4.22) coincide. Thus for any s, $0 \le s \le d(k,w)$, the equation of (4.22) with $i = \frac{k-w+1}{2} - s$ is a linear combination of other equations from the system (4.14)-(4.22), which implies the statement of the lemma for an even k. ☐

By the previous lemma Proposition 4.2 is equivalent to the fact that the system, obtained from the system (4.14)-(4.22) by crossing out the last $\left[\frac{l-w+1}{3}\right] + 1$ equations from the system (4.22), has the maximal rank. We call this system the *reduction of system* (4.14)-(4.22). For this let us show that if $x_{k-w+1-2s} = 0$ for every s such that

$$0 \le s \le d(k,w), \tag{4.29}$$

then the reduction of system (4.14)-(4.22) has the trivial solution only.

Indeed, if $x_{k-w+1} = 0$ then the first equation of (4.14) implies that $x_0 = 0$. Consequently, the first equation of (4.22) implies that $x_1 = 0$ and the second equation of (4.14) implies that $x_{k-w} = 0$. In a similar way one can show by induction that from the fact that $x_{k-w+1-2s} = 0$ for all s satisfying (4.29) it follows that $x_j = 0$ for every j such that $0 \le j \le 2d(k,w)+1$ or $k-w-2d(k,w) \le j \le k-w+1$. But then the remaining variables x_j, $2d(k,w)+2 \le j \le k-w-2d(k,w)-1$ satisfy the system (4.14)-(4.22) with k and w replaced by $\widetilde{k}$ and $\widetilde{w}$,where

$$\widetilde{k} = k - 2d(k,w) - 2, \quad \widetilde{w} = w + 2d(k,w) + 2. \tag{4.30}$$

It is easy to see that

$$\widetilde{w} > \frac{\widetilde{k}+1}{2} \quad \text{if } k \text{ is odd} \tag{4.31}$$

$$\widetilde{w} \geq \frac{\widetilde{k}}{2} \quad \text{if } k \text{ is even} \tag{4.32}$$

Not also that if k is even but $k-2w$ is not divided by 6 then the inequality (4.31) holds as well. So if $k-2w$ is not divided by 6 our proposition follows from

Lemma 4.7. *If $w > \frac{k+1}{2}$, then system* (4.14)-(4.22) *has the trivial solution only.*

Proof. By Remark 4.2 it is enough to show that in the considered case the antisymmetric matrix satisfying conditions (1), (3), and (4) of Lemma 4.1 vanishes.

As was already mentioned before, condition (3) of Lemma 4.1 implies that $c_{i,j}$ are uniquely defined by $c_{i,k+w-1-i}$ for $i = w-1, \dots, k$ (if $i+j > k+w-1$ then $c_{i,j} = 0$). In particular, we can define the mapping

$$\phi\colon (c_{w-1,k}, c_{w,k-1}, \dots, c_{k,w-1}) \mapsto (c_{0,k-w+1}, c_{1,k-w}, \dots, c_{k-w+1,0}).$$

To prove the lemma it is enough to prove that the mapping is bijective. By condition (1) of Lemma 4.1 $c_{i,j} = 0$ for $i+j < w$. If $w > \frac{k+1}{2}$, then $k-w+1 < w$. Therefore, if ϕ is bijective then $c_{i,k+w-1-i} = 0$, and consequently the whole matrix $(c_{i,j})$ vanishes as desired.

The map ϕ is the composition of the following two maps

$$\phi_1\colon (c_{w-1,k}, c_{w,k-1}, \dots, c_{k,w-1}) \mapsto (c_{0,k}, c_{1,k-1}, \dots, c_{k,0})$$

and

$$\phi_2\colon (c_{0,k}, c_{1,k-1}, \dots, c_{k,0}) \mapsto (c_{0,k-w+1}, c_{1,k-w}, \dots, c_{k-w+1,0})$$

defined inductively by recursive relations (4.10).

It is easy to see that

$$\phi_1((\underbrace{0,\dots,0}_{i}, 1, \underbrace{0,\dots,0}_{k-w+1-i})) = \left(\underbrace{0,\dots,0}_{i}, \binom{w-1}{0}, \binom{w-1}{1}, \dots, \binom{w-1}{w-1}, \underbrace{0,\dots,0}_{k-w+1-i}\right),$$

which implies that ϕ_1 is injective. Besides, ϕ_2 is surjective, because it is a composition of the maps

$$\phi_{2,s}\colon (c_{0,k-s}, c_{1,k-1-s}, \dots, c_{k-s,0}) \mapsto (c_{0,k-s-1}, c_{1,k-2-s}, \dots, c_{k-s-1,0}), \quad 0 \leq s \leq w-2$$

defined by relations (4.10) and each of this map has a one dimensional kernel and therefore surjective. Moreover, by simple induction

$$\operatorname{Ker} \phi_2 = \operatorname{span} \left\{ \left((-1)^j \binom{i+j}{i} \right)_{j=0}^{k} \right\}_{i=0}^{w-2}.$$

Finally identity (4.24) (with $l = w-1$, $m = i$ and $s = 2i$) implies that spaces $\operatorname{Im} \phi_1$ and $\operatorname{Ker} \phi_2$ are perpendicular with respect to the standard scalar product in $\mathbb{R}^{k+1}$ (in this case the right-hand side of (4.24) vanishes because $m - l = i - w + 1 < 0$). Therefore the image of ϕ_1 is transversal to the kernel of ϕ_2 .This implies that ϕ is bijective and completes the proof of the lemma. $\square$

It remains to prove the proposition in the case when $k-2w$ is divided by 6. In this case the corresponding $\widetilde{k}$ and $\widetilde{w}$ satisfy $\widetilde{w} = \frac{\widetilde{k}}{2}$ and the proposition will follow from the fact that *if w is odd and $w = \frac{k}{2}$, then system* (4.14)-(4.22) *has the trivial solution only.* For this consider the following system of equations

$$\sum_{j=0}^{2i-1} \binom{y+i}{2i+1-j} x_j = 0, \quad i = 0, 1, \dots \left[\frac{k-w}{2}\right] \tag{4.33}$$

depending on a parameter y (here $\binom{a}{b}$ is defined for any $a \in \mathbb{C}$ and integer b as usual: $\binom{a}{b} := \frac{a(a-1)\dots(a-b+1)}{b!}$ if $b \geq 0$ and $\binom{a}{b} = 0$ if $b < 0$). Note that system (4.33) coincides with system (4.22) for $y = \frac{w+1}{2}$. It can be shown that the determinant of the matrix of the system (4.14)-(4.33) is a nonzero polynomial with respect to y such that the set of its roots is the union of the following two sets: the set of all integers between $-\left[\frac{w}{4}\right]$ and $\frac{w-1}{2}$ and the set $\{-\frac{2s-1}{2} : \left[\frac{w}{4}\right] + 3 \leq s \leq w\}$. In particular, $y = \frac{w+1}{2}$ is not a root of this polynomial, which proves the last statement.

The proof of Proposition 4.2 is completed. □

To complete the proof of Theorem 4.1 it remains to prove that the set of solutions of system (4.14)-(4.22), for which the corresponding antisymmetric matrix $(c_{i,j})$ (see Remark 4.2) satisfies also condition (2) of Lemma 4.1, is an affine subspace of codimension 1 in the solution space of system (4.14)-(4.22). For this let us prove that $x_{k-w+1} \neq 0$ if and only if $c_{0,w} \neq 0$. Indeed, by definition, $x_{k-w+1} \neq 0$ is equivalent to $c_{k,w-1} \neq 0$ and hence to $c_{w-1,k} \neq 0$. Then, it is equivalent to $c_{0,k} \neq 0$ as follows from (4.10) and finally to $c_{w,0} \neq 0$ as follows from the last equation of (4.11). By the same arguments there exist a unique $c \neq 0$ such that $x_{k-w+1} = c$ if and only if $c_{0,w} = 1$. Thus $\{x_{k-w+1} = c\}$ is the affine subspace in the solution space of system (4.14)-(4.22) we are looking for. In this way the theorem is proved in the case $w \neq \frac{k+1}{2}$. It also shows that for $w = \frac{k+1}{2}$ there is at most one bi-graded Lie algebra of type (k, w) (note that $d(k, w) = 0$ in this case). On the other hand, from the constructions at the end of the Introduction, using the theory of $\mathfrak{sl}_2$-representation and Proposition 4.1 we know that there exists at least one such bi-graded Lie algebra. These proves the theorem in the case $w = \frac{k+1}{2}$ as well.
□

Corollary 4.1 implies that for $k > 2$ the unique maximally symmetric model, up to the local equivalence, for the distributions from the considered class exists if and only if $k \equiv 1 \mod 4$ or $k \not\equiv 1 \mod 4$, $d(k, 1) = 0$, and $d(k, 3) < 0$. The latter occurs exactly in the following cases: $k = 3$, 4, 6. The nontrivial products in the corresponding bi-graded Lie algebras given by (1.8) in the case $k = 3$, by (1.9) in the case $k = 4$, and by (1.10) in the case $k = 6$ can be directly obtained from conditions (1)-(4) for $c_{i,j}$ listed in Lemma 4.1. Finally, the set of maximally symmetric models is discrete and consists more than one element if and only if $d(k, 1) = 0$ and $d(k, 3) = 0$. This occurs in the case $k = 8$ only and there are exactly two nonequivalent models with 22-dimensional algebra of infinitesimal symmetries: one model with $w = 1$ and one model with $w = 3$. In all other cases the set of maximally symmetric models depend on continuous parameters.

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