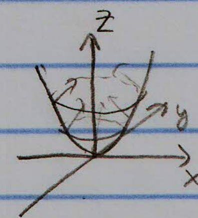
# Homework assignment #7 MATH 439 FALL 2012

Sec 3 2

Problem 8




a) $z = \underbrace{x^2+y^2}_{f(x,y)}$

Assume that the orientation is chosen such that the positive normal is directed up

The unit normal satisfies $N = \dfrac{\langle -f_x, -f_y, 1\rangle}{\sqrt{1+f_x^2+f_y^2}} = \dfrac{1}{\sqrt{1+4x^2+4y^2}}\langle -2x, -2y, 1\rangle$

Since the third component of $N$ is always positive, the image of the Gauss map lies in the upper half of the unit sphere (not including the boundary). Let us show that the image of the Gauss map is in fact equal to the upper half of the unit sphere (not including the boundary)

Take $(u,v,w) \in \underbrace{S^2}_{\text{unit sphere}}$, $w>0$ and show that there exists $x, y$ such that

$$\begin{cases} -\dfrac{2x}{\sqrt{1+4x^2+4y^2}} = u & (\text{Eq } 1) \\ -\dfrac{2y}{\sqrt{1+4x^2+4y^2}} = v & (\text{Eq } 2) \\ \dfrac{1}{\sqrt{1+4x^2+4y^2}} = w & (\text{Eq } 3) \end{cases} \quad (*)$$

Eq 3 & Eq 1: $-2xw = u \Rightarrow x = -\dfrac{u}{2w}$

Eq 2 & Eq 1 $-2yw = v \Rightarrow y = -\dfrac{v}{2w}$

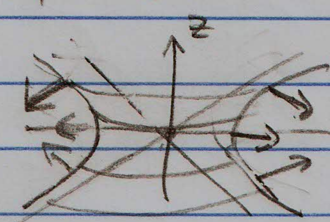
Since $u^2+v^2+w^2=1$ substituting these x and y to Eq 3 we obtain the identity $\Rightarrow$ these x and y are indeed solutions of (*) $\Longrightarrow$

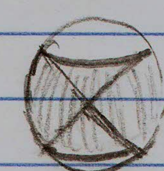
**Answer** Upper hemisphere without the equator

(lower hemisphere if we take an opposite orientation

Sec 3.2 Problem 8

b) $x^2+y^2-z^2=1$



$$N = \frac{\langle 2x, 2y, -2z\rangle}{2\sqrt{x^2+y^2+z^2}} = \frac{\langle x, y, -z\rangle}{\sqrt{x^2+y^2+z^2}}$$

if the orientation is chosen with $N$ directed outward

Let us find $(u,v,w)\in S^2$ s.t. the system

$$\begin{cases} \dfrac{x}{\sqrt{x^2+y^2+z^2}} = u & (\text{Eq } 1') \\[2ex] \dfrac{y}{\sqrt{x^2+y^2+z^2}} = v & (\text{Eq } 2') \qquad (**) \\[2ex] -\dfrac{z}{\sqrt{x^2+y^2+z^2}} = w & (\text{Eq } 3') \end{cases}$$

has a solution $(x,y,z)$ such that $x^2+y^2-z^2=1$

$$x^2+y^2 = 1+z^2 \overset{(\text{Eq } 3')}{\Longrightarrow} -\frac{z}{\sqrt{2z^2+1}} = w \Longrightarrow \frac{z^2}{2z^2+1} = w^2$$

Hence $z^2 = 2z^2w^2 + w^2 \iff z^2(1-2w^2) = w^2 \iff z^2 = \dfrac{w^2}{1-2w^2} \geq 0$

Therefore $1-2w^2>0 \Longrightarrow w^2<\frac{1}{2}$ or equivalently $|w|<\frac{1}{\sqrt{2}}$

Thus the equation $-\dfrac{z}{\sqrt{2z^2+1}} = w$ (w.r.t $z$) has a solution

if and only if $|w|<\frac{1}{\sqrt{2}}$. Then from (Eq 1') & (Eq 2')

$$x = u\sqrt{1+2z^2} = u\sqrt{1+\frac{2w^2}{1-2w^2}} = \frac{u}{\sqrt{1-2w^2}}$$

$$y = v\sqrt{1+2z^2} = \frac{v}{\sqrt{1-2w^2}}$$

So the system (**) has a solution $\iff |w| < \frac{1}{\sqrt{2}}$

Answer: $\left\{ (u,v,w) : \begin{array}{l} u^2+v^2+w^2=1 \\ |w| < \frac{1}{\sqrt{2}} \end{array} \right\}$

Sec. 3.2

Problem 10 Let $\alpha(s)$ be a parametrization of $C$ by an arc length. The normal to the osculating plane is the binormal $b(s)$. The statement of the exercize is equivalent to the fact that

If ~~$f(s) := \langle N(\alpha(s)), b(s)\rangle$ is constant or, equivalently,~~

if $f'(s) \equiv 0$, then $C$ is a plane curve.

Calculate $f'(s)$:

$$f'(s) = \left\langle \frac{d}{ds} N(\alpha(s)), b(s) \right\rangle + \left\langle N(\alpha(s)), \frac{d}{ds} b(s) \right\rangle \qquad (***)$$

~~Since $C$ is the line of curvature then~~

$$\frac{d}{ds} N(\alpha(s)) = dN_{\alpha(s)}\alpha'(s) \parallel \alpha'(s) \quad \text{(Rodrigues Thm)}$$

Since $\underbrace{\alpha'(s)}_{t(s)} \perp b(s)$ this implies that

the first term in (***) vanishes

Therefore,

$$f'(s) = \langle N(\alpha(s)), b'(s) \rangle = \langle N(\alpha(s)), \tau(s) n(s) \rangle =$$

$$= \tau(s) \langle N(\alpha(s)), n(s) \rangle \Rightarrow$$

$$f'(s) = 0 \Rightarrow \tau(s) \underbrace{\langle N(\alpha(s)), n(s) \rangle}_{\text{normal curvature in the direction } \alpha'(s)} = 0 \quad (4)$$

Since $C$ is nowhere tangent to an asymptotic direction

$$\langle N(\alpha(s)), n(s) \rangle \neq 0 \quad \forall s \overset{(4)}{\Rightarrow} \tau(s) \equiv 0 \Rightarrow$$

$C$ is a plane curve, q.e.d.

Sec 3.2

Problem 11. Assume that $w$ and $w'$ are unit vectors in the conjugate directions $r$ and $r'$ respectively. Here we also <u>must</u> make additional assumption that $p$ <u>is not an umbilical point</u> (otherwise any two orthogonal directions are conjugate and the statement of the exercize is wrong)

Let $e_1$ and $e_2$ are the principal directions corresponding to $k_1 > k_2$ and assume that

$$w = \cos\theta\, e_1 + \sin\theta\, e_2$$

$$w' = \cos\varphi\, e_1 + \sin\varphi\, e_2$$

Then $\langle dN_p w, w' \rangle = 0 \Leftrightarrow$

$$k_1 \cos\theta \cos\varphi + k_2 \sin\theta \sin\varphi = 0$$

Let $\alpha$ be the angle between $\theta$ and $\varphi$, $\varphi = \theta + \alpha$

$\Downarrow$

$$k_1 \cos\theta \cos(\theta+\alpha) + k_2 \sin\theta \sin(\theta+\alpha) = 0 \iff$$

$$k_1 (\cos^2\theta \cos\alpha - \cos\theta \sin\theta \sin\alpha) + k_2 (\sin^2\theta \cos\alpha + \sin\theta \cos\theta \sin\alpha) =$$

$$= (k_1 \cos^2\theta + k_2 \sin^2\theta) \cos\alpha + (k_2 - k_1) \cos\theta \sin\theta \sin\alpha = 0 \Rightarrow$$

$$\operatorname{tg}\alpha = \frac{k_1 \cos^2\theta + k_2 \sin^2\theta}{(k_1 - k_2)\cos\theta \sin\theta} = \frac{k_2}{k_1 - k_2} \tan\theta + \frac{k_1}{k_1 - k_2} \operatorname{cotan}\theta$$

Let us show that the function

$$g(\theta) := \left| \frac{k_2}{k_1 - k_2} \tan\theta + \frac{k_1}{k_1 - k_2} \operatorname{cotan}\theta \right| \text{ attains}$$

its minimum on $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ exactly at two points symmetric w.r.t. $\theta = 0$. This will imply the statement.

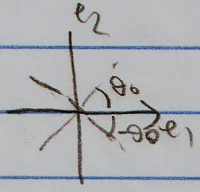
1) Case $0 < \theta < \frac{\pi}{2}$: $g(\theta) > 0$, $\lim_{\theta \to 0+} g(\theta) = \lim_{\theta \to \frac{\pi}{2}-} g(\theta) = +\infty$ (this corresponds to the fact that the principal directions are conjugate) $\Rightarrow$ minimum is attained. Further,

$$g'(\theta) = \frac{k_2}{k_1 - k_2} \frac{1}{\cos^2\theta} - \frac{k_1}{k_1 - k_2} \frac{1}{\sin^2\theta} = 0 \Rightarrow$$

$$\tan^2\theta = \frac{k_1}{k_2} \Rightarrow \left(\text{since } 0 < \theta < \frac{\pi}{2}\right) \operatorname{tg}\theta = \sqrt{\frac{k_1}{k_2}} \Rightarrow$$

unique critical point $\theta_0$ and it must be a minimum (you can argue also by showing that $g'(\theta) < 0$ on $0 < \theta < \theta_0$ and $g'(\theta) > 0$ on $\theta_0 < \theta < \frac{\pi}{2}$)

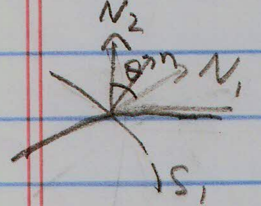
2) Case $-\frac{\pi}{2} < \theta < 0$ $\rightarrow$ the same conclusion with the minimum obtained at $-\theta_0$ because $g(\theta) = g(-\theta)$.




Rem Note that if $\theta$ is a point of minimum of $g$ then $\theta + \alpha$ is also a point of minimum of $g$.

## Section 3.2

### Problem 14




Assume that $\theta_1$ and $\theta_2$ are the angles between $n$ and $N_1$ and $n$ and $N_2$, respectively

$$\Downarrow$$

$$\theta = \theta_1 + \theta_2$$

$$\lambda_1 = k\cos\theta_1$$

$$\lambda_2 = k\cos\theta_2$$

Therefore RHS of the require identity has the form

$$\lambda_1^2 + \lambda_2^2 - 2\lambda_1\lambda_2\cos\theta = k^2\cos^2\theta_1 + k^2\cos^2\theta_2 - 2k^2\cos\theta_1\cos\theta_2\cos(\theta_1+\theta_2) =$$

$$= k^2\left(\cos^2\theta_1 + \cos^2\theta_2 - 2\cos\theta_1\cos\theta_2\left(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\right)\right) =$$

$$= k^2\left(\cos^2\theta_1 + \cos^2\theta_2 - 2\cos^2\theta_1\cos^2\theta_2 + 2\cos\theta_1\cos\theta_2\sin\theta_1\sin\theta_2\right) =$$

$$= k^2\left(\cos^2\theta_1\underbrace{(1-\cos^2\theta_2)}_{\sin^2\theta_2} + \cos^2\theta_2\underbrace{(1-\cos^2\theta_1)}_{\sin^2\theta_1} + 2\cos\theta_1\cos\theta_2\sin\theta_1\sin\theta_2\right)$$

$$= k^2\left(\underbrace{\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2}_{\sin(\theta_1+\theta_2)}\right)^2 = k^2\sin^2\theta \qquad \text{q.e.d.}$$

## Section 3.3

### Problem 1

$$f(x,y) = axy$$

By the formula derived in class (see also p. 163)

since at the origin $f_x = f_y = 0$, we have

$$K = f_{xx}f_{yy} - (f_{xy})^2 = -a^2$$

$$H = f_{xx} + f_{yy} = 0 \qquad \text{q.e.d}$$

## Problem 5

$$\mathbf{x}(u,v) = \left(u - \frac{u^3}{3} + uv^2,\ v - \frac{v^3}{3} + vu^2,\ u^2 - v^2\right)$$

a) $$x_u = (1-u^2+v^2,\ 2uv,\ 2u)$$
$$x_v = (2uv,\ 1-v^2+u^2,\ -2v)$$

$$E = \langle x_u, x_u\rangle = (1-u^2+v^2)^2 + 4u^2v^2 + 4u^2 = 1+u^4+v^4-2u^2+$$
$$+2v^2-2u^2v^2+4u^2v^2+4u^2 = 1+u^4+v^4+2u^2+2v^2+2u^2v^2 = (1+u^2+v^2)^2$$

$$F = \langle x_u, x_v\rangle = 2(\underbrace{1-u^2+v^2+1-v^2+u^2}_{2})uv - 4uv = 4uv-4uv=0$$

$$G = \langle x_v, x_v\rangle = 4u^2v^2 + (1-v^2+u^2)^2 + 4v^2 = (1+u^2+v^2)^2$$

$\Downarrow$

$$E = G = (1+u^2+v^2)^2,\quad F=0$$

Problem 5

b) $$N = \frac{x_u \times x_v}{|x_u \times x_v|}$$

$$x_u \times x_v = \begin{vmatrix} i & j & k \\ 1-u^2+v^2 & 2uv & 2u \\ 2uv & 1-v^2+u^2 & -2v \end{vmatrix} =$$

$$= (-4uv^2 - 2u + 2uv^2 - 2u^3)\hat{i} - (-2v + 2u^2v - 2v^3 - 4u^2v)\hat{j} +$$
$$\left((1-(u^2-v^2))(1+u^2-v^2) - 4u^2v^2\right)\hat{k} = -2u(u^2+v^2+1)\hat{i} + 2v(u^2+v^2+1)\hat{i}$$
$$+ \left(\underbrace{1-(u^2-v^2)^2 - 4u^2v^2}_{1-(u^2+v^2)^2 = (1+u^2+v^2)(1-u^2-v^2)}\right)\hat{k} = (u^2+v^2+1)(-2u,\ 2v,\ 1-u^2-v^2)$$

$$\|x_u \times x_v\| = (u^2+v^2+1)\sqrt{\underbrace{4u^2+4v^2+(1-u^2-v^2)^2}_{(1+u^2+v^2)^2}} = (1+u^2+v^2)^2 \Rightarrow$$

$$N = \frac{1}{1+u^2+v^2}(-2u,\ 2v,\ 1-u^2-v^2)$$

$x_{uu} = \langle -2u, 2v, 2 \rangle$

$x_{uv} = \langle 2v, 2u, 0 \rangle$

$x_{vv} = \langle 2u, -2v, -2 \rangle$

$\Downarrow$

$$e = \langle N, x_{uu} \rangle = \frac{1}{1+u^2+v^2} \Big( \underbrace{4u^2+4v^2+2-2u^2-2v^2}_{2+2u^2+2v^2} \Big) = 2$$

$$f = \langle N, x_{uv} \rangle = \frac{1}{1+u^2+v^2} (4uv - 4uv) = 0$$

$$g = \langle N, x_{vv} \rangle = \frac{1}{1+u^2+v^2} \left( -4u^2 - 4v^2 - 2 + 2u^2 + 2v^2 \right) = -2$$

Problem 5

(c) Since $F = f = 0$

$$k_1 = \frac{e}{E} = \frac{2}{(1+u^2+v^2)^2}$$

$$k_2 = \frac{f}{F} = -\frac{2}{(1+u^2+v^2)^2}$$

d) The lines of curvature are coordinate curves because $F = f = 0$

c) $\mathrm{II} = du^2 - dv^2 = 0 \Rightarrow$ if $\alpha(t) = x(u(t), v(t))$ is an asymptotic line then

$$u'^2 - v'^2 = 0 \Leftrightarrow (u' - v')(u' + v') = 0 \Rightarrow$$

$u - v = \text{const}$ or

$u + v = \text{const}$ q.e.d.