MATH 439 Diff. geom. of curves and surfaces Fall 2012

Homework assignment 1 solution

Section 1.2

Problem 3 Let $\alpha(t) = (x_1(t), \dots, x_n(t))$

$\alpha''(t) = 0 \Leftrightarrow x_i''(t) = 0$ for any $i \Leftrightarrow x_i'(t) \equiv a_i \Leftrightarrow$

$x_i(t) = a_i t + b \Rightarrow \alpha(t)$ is either a straight line

(if $\sum_{i=1}^{n} a_i^2 > 0$) or a point.

Problem 5 $|\alpha(t)| \equiv C \Leftrightarrow |\alpha(t)|^2 \equiv C^2 \Leftrightarrow \langle \alpha(t), \alpha(t) \rangle = C \Leftrightarrow$
$(C>0)$

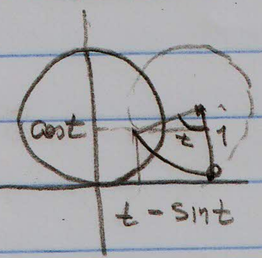
(after differentiation) $\langle \alpha'(t), \alpha(t) \rangle = 0 \Leftrightarrow \alpha'(t) \perp \alpha(t)$

($C \neq 0$ because of the assumption that $\alpha'(t) \neq 0$)

Section 1.3

Problem 2

a)



$\alpha(t) = (t - \sin t, \cos t)$ (see the picture)

Determine its singular points

$\alpha'(t) = (1 - \cos t, \sin t) = 0 \Leftrightarrow$

$\begin{cases} \cos t = 1 \\ \sin t = 0 \end{cases} \Leftrightarrow \boxed{t = 2\pi k, k \in \mathbb{Z}}$

b) Complete rotation of the disk corresponds to $0 \le t \le 2\pi$

$\text{length} = \int_0^{2\pi} |\alpha'(t)|\,dt = \int_0^{2\pi} \sqrt{(1-\cos t)^2 + \sin^2 t}\,dt =$

$= \int_0^{2\pi} \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t}\,dt = \int_0^{2\pi} \sqrt{2 - 2\cos t}\,dt = 2\int_0^{2\pi} \sin\frac{t}{2}\,dt =$

$= 2 \cdot \frac{1}{\frac{1}{2}} \left(-\cos\frac{t}{2}\right)\Big|_0^{2\pi} = 2 \cdot 2 \cdot (1 - (-1)) = \boxed{8}$

Problem 4

a) $\alpha' = \left(\cos t, -\sin t + \frac{1}{\tan\frac{t}{2}} \cdot \frac{1}{\cos^2\frac{t}{2}} \cdot \frac{1}{2}\right) =$

$= \left(\cos t, \quad -\sin t + \frac{1}{2\sin\frac{t}{2}\cos\frac{t}{2}}\right) = \left(\cos t, -\sin t + \frac{1}{\sin t}\right) \Rightarrow$

α has derivatives of all orders on $(0,\pi) \Rightarrow \alpha$ is differentiable on $(0,\pi)$. Let us check if it is regular

$$\alpha'(t) = 0 \Leftrightarrow \begin{cases} \cos t = 0 \\ -\sin t + \frac{1}{\sin t} = 0 \end{cases} \Rightarrow t = \frac{\pi}{2} \text{ and } t = \frac{\pi}{2} \text{ also satisfies the second equation}$$

$\Rightarrow t = \frac{\pi}{2}$ is the unique singular point

b) The equation of the tangent line at $t = t_0$ is

$$x(\tau) = \sin t_0 + \cos t_0 \, \tau$$
$$y(\tau) = \cos t_0 + \log \tan\frac{t_0}{2} + \left(-\sin t_0 + \frac{1}{\sin t_0}\right)\tau$$

(here τ is a parameter on the tangent line)

To find the parameter τ for which the corresponding point of the tangent line intersects the y-axis we should solve the equation

$$x(\tau) = 0 \Leftrightarrow$$
$$\sin t_0 + \cos t_0 \, \tau = 0 \Leftrightarrow \tau = -\tan(t_0) \Rightarrow$$ the require segment of the tangent line corresponds to $\tau \in (-\tan t_0, 0) \Rightarrow$ its length $=$

$$= \int_{-\tan t_0}^{0} |\alpha'(t_0)| \, dt = \sqrt{\cos^2 t_0 + \left(-\sin t_0 + \frac{1}{\sin t_0}\right)^2} \tan t_0 =$$

$$\sqrt{\cos^2 t_0 + \sin^2 t_0 - 2 + \frac{1}{\sin^2 t_0}} \tan t_0 = \sqrt{\frac{1}{\sin^2 t_0} - 1} \tan t_0 = \cot t_0 \tan t_0 = 1$$