Fall 2012

# Assignment 3 in Differential Geometry of curves and surfaces (MATH 439) Solutions

**Problem 1** $\tau(t) = -\dfrac{(\alpha' \times \alpha'') \cdot \alpha^{(3)}}{|\alpha' \times \alpha''|^2}$

$\alpha(t) = (e^t, e^{-t}, \sqrt{2}t)$

$\alpha'(t) = (e^t, -e^{-t}, \sqrt{2})$

$\alpha''(t) = (e^t, e^{-t}, 0)$

$\alpha^{(3)}(t) = (e^t, -e^{-t}, 0)$

$$\alpha' \times \alpha'' = \begin{vmatrix} i & j & k \\ e^t & -e^{-t} & \sqrt{2} \\ e^t & e^{-t} & 0 \end{vmatrix} = -\sqrt{2}e^{-t}\hat{i} + \sqrt{2}e^t \hat{j} + (1+1)\hat{k} =$$

$= (-\sqrt{2}e^{-t}, \sqrt{2}e^t, 2)$

$(\alpha' \times \alpha'') \cdot \alpha^{(3)} = -\sqrt{2} - \sqrt{2} = -2\sqrt{2}$

$|\alpha' \times \alpha''|^2 = 2e^{-2t} + 2e^{2t} + 4 = 2(e^t + e^{-t})^2$

$\Downarrow$

$$\tau(t) = \frac{\cancel{2}\sqrt{2}}{\cancel{2}(e^t+e^{-t})^2} = \boxed{\frac{\sqrt{2}}{(e^t+e^{-t})^2}}$$

**Section 1.5 Problem 10**

a) $\alpha$ has all derivatives for $t \neq 0$

For $t=0$: $\lim\limits_{t \to 0+} \dfrac{\alpha(t) - \alpha(0)}{t} = \left(1, 0, \lim\limits_{t\to 0+} \dfrac{e^{-\frac{1}{t^2}}}{t}\right) = (1,0,0)$

Indeed

$\lim\limits_{t \to 0+} \dfrac{e^{-\frac{1}{t^2}}}{t} = \lim\limits_{y \to +\infty} y e^{-y^2} = 0$; similarly $\lim\limits_{t\to 0-} \dfrac{\alpha(t)-\alpha(0)}{t} = \left(1, \lim\limits_{t \to 0-} \dfrac{e^{-\frac{1}{t^2}}}{t}, 0\right) =$

$= (1,0,0)$

$\Rightarrow \alpha$ is differentiable at $t=0$ and $\alpha'(0)=(1,0,0)$

Similarly one can show that $\alpha$ has all derivatives at $0$

b) $\alpha'(0) \neq 0$

$\alpha'(t) = \left(1, 0, \frac{2}{t^3} e^{-\frac{1}{t^2}}\right)$ if $t>0$

$\alpha'(t) = \left(1, \frac{2}{t^3} e^{-\frac{1}{t^2}}, 0\right)$ if $t>0$

In both case $\alpha'(t) \neq 0 \Rightarrow \alpha(t)$ is regular for all $t$

c) $\alpha''(t) = \left(0, 0, -\frac{6}{t^4} e^{-\frac{1}{t^2}} + \frac{4}{t^6} e^{-\frac{1}{t^2}}\right) =$

$= \left(0, 0, \frac{2}{t^6}(2-3t^2) e^{-\frac{1}{t^2}}\right)$ if $t>0$

Similarly $\alpha''(t) = \left(0, \frac{2}{t^6}(2-3t^2) e^{-\frac{1}{t^2}}, 0\right)$ if $t<0$

$\alpha''(0) = 0$. Indeed

$$\lim_{t \to 0+} \frac{\alpha'(t)-\alpha'(0)}{t} = \left(0, 0, \lim_{t \to 0+} \frac{2}{t^4} e^{-\frac{1}{t^2}}\right) = (0,0,0)$$

Similarly $\lim_{t \to 0-} \frac{\alpha'(t)-\alpha'(0)}{t} = \left(0, \lim_{t \to 0-} \frac{2}{t^4} e^{-\frac{1}{t^2}}, 0\right) = (0,0,0)$

If $t>0$ $\alpha'(t) \times \alpha''(t) = \begin{vmatrix} i & j & k \\ 1 & 0 & \frac{2}{t^3} e^{-\frac{1}{t^2}} \\ 0 & 0, & \frac{2}{t^6}(2-3t^2) e^{-\frac{1}{t^2}} \end{vmatrix} =$

$\frac{2}{t^6}(2-3t^2) e^{-\frac{1}{t^2}} \hat{k} \Rightarrow |\alpha'(t) \times \alpha''(t)| = \frac{2}{t^6} |2-3t^2| e^{-\frac{1}{t^2}}$

$|\alpha'(t)| = \sqrt{1 + \frac{4}{t^6} e^{-\frac{2}{t^2}}} \Rightarrow$

$$k(t) = \frac{\frac{2}{t^6}\,|2-3t^2|\,e^{-\frac{1}{t^2}}}{\sqrt{1+\frac{4}{t^6}e^{-\frac{2}{t^2}}}}$$

Note that by analogous computation the same formula holds for $t<0 \Rightarrow$

$\Rightarrow$ If $t \neq 0$ then $k(t)=0 \Leftrightarrow 2-3t^2=0 \Leftrightarrow t=\pm\sqrt{\frac{2}{3}}$

If $t=0$ $\alpha''(0)=0 \Rightarrow k(0)=0$

c) Not that for $t>0$ the osculating plane is span $\alpha'(t), \alpha''(t)$ is $y=0$
and for $t<0$ it is $z=0$, which implies the same answers for the corresponding one-sided limits

d) If $t>0$ then the curve lies in the plane $y=0 \Rightarrow$
$\tau(t)=0,\ t>0$
If $t<0$ then the curve lies in the plane $z=0 \Rightarrow \tau(t)=0,\ t<0$

$\Rightarrow \tau(0)$ can be defined equal to zero by passing to the limit as $t \to 0$

Section 1.5



Problem 15

Indeed

$|\tau(s)| = |b'(s)| \Rightarrow |\tau(s)|$ is determined by $b(s)$

$b'(s) = \tau(s)\, n(s) \Rightarrow$

$b''(s) = \tau'(s) n(s) + \tau(s)\left(-k(s) t(s) - \tau(s) b(s)\right) =$

$= \tau(s)\, n(s) - k(s)\tau(s)\, t(s) - \tau^2(s)\, b(s) \Rightarrow$

$$b''(s) \times b'(s) = -k(s)\tau^2(s) \underbrace{t(s)\times n(s)}_{b(s)} - \tau^3(s) \underbrace{n(s)\times b(s)}_{t(s)} \Rightarrow$$

$$\left(b''(s)\times b'(s)\right)\cdot b(s) = -k(s)\tau^2(s) \Rightarrow$$

since $\tau^2(s)$ is determined by $b(s) \Rightarrow k(s)$ is determined by $b(s)$. More precisely

$$k(s) = -\frac{\left(b''(s)\times b'(s)\right)\cdot b(s)}{|b'(s)|^2}$$

Section 1.5 Problem 17 Assume that $\alpha$ is parametrized by arclength

a) $\alpha$ is a helix $\Leftrightarrow$ there exists a unit vector $v$ such that

$\langle \underbrace{\alpha'(s)}_{t(s)}, v\rangle = \text{const} \Leftrightarrow$

$\langle \alpha''(s), v\rangle = 0 \Leftrightarrow v \perp n(s) \Leftrightarrow$

$v = c_1 t(s) + \gamma(s) b(s)$ for some constant $c_1 \Rightarrow$

$\gamma(s)$ is constant because $v$ is unit: $\gamma(s) \equiv c_2$ s.t $c_1^2 + c_2^2 = 1$

Differentiating the identity
$v = c_1 t(s) + c_2 b(s)$ and using the structure eq for
Frenet frame: $0 = (c_1 k(s) + c_2 \tau(s))\, n(s) \Rightarrow$
$c_1 k(s) + c_2 \tau(s) = 0 \Rightarrow \frac{k(s)}{\tau(s)} = -\frac{c_2}{c_1}$

In the opposite direction, if
$\frac{k(s)}{\tau(s)} = C$, take $\theta$ s.t $\tan\theta = -C \Rightarrow$

$\cos\theta\, k(s) + \sin\theta\, \tau(s) = 0 \Rightarrow$

the vector $v = \cos\theta\, t(s) + \sin\theta\, b(s)$ is independent of $s$
Besides, $v \perp n(s) \Rightarrow \langle v, \alpha''(s)\rangle = 0 \Rightarrow \langle v, \alpha'(s)\rangle$ is constant

b) $\alpha$ is a helix $\Leftrightarrow$ there exist a unit vector $v$ s.t $\langle v, \alpha'(s)\rangle = \text{const} \Leftrightarrow \langle v, \alpha''(s)\rangle = 0 \Leftrightarrow v \perp n(s) \Leftrightarrow$
the line containing $n(s)$ (and passing through $\alpha(s)$) is parallel to the plane orthogonal to $v$

c) $\alpha$ is a helix $\overset{\text{(item a)}}{\Leftrightarrow}$ there exist a unit vector $v$ that $v = c_1 t(s) + c_2 b(s)$ for some constant $c_1$ and $c_2$
$\Rightarrow \langle v, b(s)\rangle \equiv c_2$
If in opposite direction $\langle v, b(s)\rangle = \text{const}$ then
$\langle v, b'(s)\rangle = 0 \Leftrightarrow \langle v, n(s)\rangle = 0 \overset{\text{item b)}}{\Longrightarrow} \alpha$ is a helix

d) Let $v = (0,0,1)$

then $\langle \alpha'(s), v\rangle = \frac{b}{c} = \text{const} \Rightarrow$ (by item a)

$\alpha$ is a helix

Let us find $\frac{k}{\tau}$

1) $\alpha$ is parametrized by arclength

Indeed

$$\alpha'(s) = \left(\frac{a}{c}\sin\theta(s), \frac{a}{c}\cos\theta(s), \frac{b}{c}\right) \Rightarrow$$

$$\alpha'(s) = \sqrt{\frac{a^2}{c^2} + \frac{b^2}{c^2}} = 1$$

$$\alpha''(s) = \left\langle \frac{a}{c}\cos\theta(s)\theta'(s), -\frac{a}{c}\sin\theta(s)\theta'(s), 0\right\rangle \Rightarrow$$

$$k(s) = |\alpha''(s)| = \boxed{\frac{a}{c}\theta'(s)}$$

$$n(s) = \langle \cos\theta(s), -\sin\theta(s), 0)$$

$$b(s) = t(s)\times n(s) = \begin{vmatrix} i & j & k \\ \frac{a}{c}\sin\theta(s) & \frac{a}{c}\cos\theta(s), & \frac{b}{c} \\ \cos\theta(s) & -\sin\theta(s) & 0 \end{vmatrix} =$$

$$= \left\langle \frac{b}{c}\sin\theta(s), \frac{b}{c}\cos\theta(s), -\frac{a}{c}\right)$$

$$b'(s) = \left\langle \frac{b}{c}\cos\theta(s)\theta'(s), -\frac{b}{c}\sin\theta(s)\theta'(s), 0\right\rangle = \frac{b}{c}\theta'(s)\, n(s)$$

$$\Rightarrow \boxed{\tau(s) = \frac{b}{c}\theta'(s)} \Rightarrow \boxed{\frac{k(s)}{\theta(s)} = \frac{a}{b}}$$

## Section 1.6

### Problem 1

Assume that v is a vector orthogonal to the plane P. Then condition 1 implies that

$$\langle v, \alpha'(s)\rangle = 0$$

On the other hand condition 2 implies that

$$\langle v, \alpha''(s)\rangle = 0$$

Indeed, let $f(s) := \langle v, \alpha(s)\rangle$

Then $f'(s) = \langle v, \alpha'(s)\rangle = 0$ and $f''(s) = \langle v, \alpha''(s)\rangle$

If $f''(s) \neq 0$ then s is a point of local extremum of $f(s)$, which contradicts condition 2 $\Rightarrow$ $f''(s) = 0 \Leftrightarrow \langle v, \alpha''(s)\rangle = 0 \Rightarrow$

$v \perp \text{span}(\alpha'(s), \alpha''(s)) \Leftrightarrow \Pi = \text{span}(\alpha'(s), \alpha''(s))$,

i.e. $\Pi$ is the osculating plane.

**Problem 3** Assume that $\alpha$ is parametrized by arc length parameter $\alpha$. Note that s is no longer an arc length parameter for $\pi \circ \alpha$. However $(\pi \circ \alpha)'(s) = \alpha'(s)$ and $(\pi \circ \alpha)''(s) = \alpha''(s)$ just by the definition of osculating plane $\Rightarrow$ by problem 12 b

the curvature of $\pi \circ \alpha = \dfrac{|(\pi\circ\alpha)'(s) \times (\pi\circ\alpha)''(s)|}{|(\pi\circ\alpha)'(s)|^3} =$

$= \dfrac{|\alpha'(s) \times \alpha''(s)|}{|\alpha'(s)|} = |\alpha''(s)| =$ the curvature of $\alpha$

(because $|\alpha'(s)| = 1$ and $\alpha'(s) \perp \alpha''(s)$)