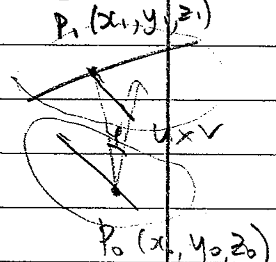
Home assignment 2 solution Math 439

Section 1.4

(5) $p \in$ the plane passing through $p_1, p_2, p_3 \iff$ vectors $p-p_1, p-p_2, p-p_3$ are parallel to this plane $\iff$ the volume of the parallelepiped spanned by $p-p_1, p-p_2, p-p_3$ is $0 \iff ((p-p_1)\times(p-p_2))\cdot p-p_3 = 0$

(8)



The distance ρ = the distance between two planes passing through $P_0(x_0, y_0, z_0)$ and $P_1(x_1, y_1, z_1)$ and orthogonal to $u\times v$ $= |\overrightarrow{P_0P_1}|\cos\varphi$, where φ is the angle between $\overrightarrow{P_0P_1}$ and $u\times v$ = (see the figure)

$$= \frac{(u\times v)\cdot\overrightarrow{P_0P_1}}{|u\times v|} = \frac{(u\times v)\cdot r}{|u\times v|} \quad \text{q.e.d}$$

Section 1.5

(7) a) Without loss of generality we can assume that α is parametrized by arclength s.

$$\beta'(s) = \alpha'(s) + \left(\frac{1}{k(s)}\right)' n(s) + \frac{1}{k(s)} n'(s) \doteq (*)$$

$\alpha'(s) = t(s)$ and from the Frenet equation

$n'(s) = -k(s)t(s) \Rightarrow$

$$(*) = t(s) + \left(\frac{1}{k(s)}\right)' n(s) + \frac{1}{k(s)}\left(-k(s)n(s)\right) =$$

$$= \left(\frac{1}{k(s)}\right)' n(s) \Rightarrow \beta'(s) \parallel n(s) \Rightarrow \beta'(s) \perp \alpha'(s)$$

q.e.d.

b) To find the point of intersection of the normals to α at s_1 and s_2 we have to solve the following equation

$$\alpha(s_1) + \tau_1 n(s_1) = \alpha(s_2) + \tau_2 n(s_2)$$

w.r.t. τ_1 and τ_2 $\Leftrightarrow$

$$\tau_1 n(s_1) - \tau_2 n(s_2) = \alpha(s_2) - \alpha(s_1)$$

This is actually the system of 2 linear equations w.r.t τ_1 and τ_2: If $n(t) = (n_1(t), n_2(t))$, $\alpha(t) = (x(t), y(t))$

then
$$\begin{cases} \tau_1 n_1(s_1) - \tau_2 n_1(s_2) = x(s_2) - x(s_1) \\ \tau_1 n_2(s_1) - \tau_2 n_2(s_2) = y(s_2) - y(s_1) \end{cases}$$

$\Downarrow$ By Cramer rule

$$\tau_2 = - \frac{\begin{vmatrix} n_1(s_1) & x(s_2) - x(s_1) \\ n_2(s_1) & y(s_2) - y(s_1) \end{vmatrix}}{\begin{vmatrix} n_1(s_1) & n_1(s_2) \\ n_2(s_1) & n_2(s_2) \end{vmatrix}} =$$

$$= \frac{\begin{vmatrix} x(s_2) - x(s_1) & n_1(s_1) \\ y(s_2) - y(s_1) & n_2(s_1) \end{vmatrix}}{\begin{vmatrix} n_1(s_1) & n_1(s_2) - n_1(s_1) \\ n_2(s_1) & n_2(s_2) - n_2(s_1) \end{vmatrix}} \xrightarrow[\text{(using Mean Value Thm)}]{s_2 \to s_1} \frac{\begin{vmatrix} x'(s_1) & n_1(s_1) \\ y'(s_1) & n_2(s_1) \end{vmatrix}}{\begin{vmatrix} n_1(s_1) & n_1'(s_1) \\ n_2(s_1) & n_2'(s_1) \end{vmatrix}} =$$

we again ~~work with~~ natural parameter

$$= \frac{\begin{vmatrix} \mathbf{t}_1(s_1) & n_1(s_1) \\ \mathbf{t}_2(s_1) & n_2(s_1) \end{vmatrix}}{\begin{vmatrix} n_1(s_1) & -k(s_1)\mathbf{t}_1(s_1) \\ n_2(s_1) & -k(s_1)\mathbf{t}_1(s_1) \end{vmatrix}} = \frac{1}{k(s_1)} \Rightarrow \tau_2 \xrightarrow[s_2 \to s_1]{} \frac{1}{k(s_1)}$$

q.e.d.

(8) a) As proved in class the signed curvature of the curve $\alpha(t) = (t, f(t))$ is given by

$$k(t) = \frac{f''(t)}{(1+f'(t)^2)^{3/2}}$$

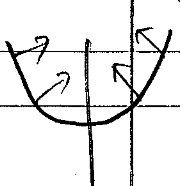
In our case $f(t) = \cosh t \Rightarrow k(t) = \frac{\cosh t}{(1+\sinh^2 t)^{3/2}} = \frac{\cosh t}{\cosh^3 t} = \frac{1}{\cosh^2 t}$

Rem: Since $k(t) > 0$ the signed curvature = curvature

b) Rem: First, the common mistake here is that people claim that $n(t) = \frac{\alpha''(t)}{|\alpha''(t)|}$ $\left(\text{or } \frac{\alpha''(t)}{k(t)}\right)$. It is not true because t is not a natural parameter ($|\alpha'(t)| \neq 1$) and $\alpha''(t) \not\perp \alpha'(t)$.

So $\alpha'(t) = (1, \sinh t) \Rightarrow$ $|\alpha'(t)| = \cosh t$ (using the fact that given a vector (a, b) the unit orthogonal vector to it is $\pm \frac{1}{\sqrt{a^2+b^2}}(b, -a)$) $\Rightarrow n(t)$ is either $\frac{1}{\cosh t}(\sinh t, -1)$ or $\frac{1}{\cosh t}(-\sinh t, 1)$. We choose the right direction of n using the fact that $\langle \alpha''(t), n(t) \rangle > 0$, i.e the angle between $\alpha''(t)$ and $n(t)$ has to be acute.

$\alpha''(t) = (0, \cosh(t)) \Rightarrow n(t) = \frac{1}{\cosh t}(-\sinh t, 1)$



Substituting this to the formula $\beta(t) = \alpha(t) + \frac{1}{k(t)} n(t)$ we get the required $\square$

11 a) $x(\theta) = \rho(\theta)\cos\theta \Rightarrow x'(\theta) = \rho'(\theta)\cos\theta - \rho(\theta)\sin\theta$

$y(\theta) = \rho(\theta)\sin\theta \quad\;\; y'(\theta) = \rho'(\theta)\sin\theta + \rho(\theta)\cos\theta$

$\Downarrow$

$$x'(\theta)^2 + y'(\theta)^2 = \left(\rho'(\theta)\cos\theta - \rho(\theta)\sin\theta\right)^2 + \left(\rho'(\theta)\sin\theta + \rho(\theta)\cos\theta\right)^2 =$$

$$= \rho'(\theta)^2\cos^2\theta - \cancel{2\rho'(\theta)\rho(\theta)\cos\theta\sin\theta} + \rho(\theta)^2\sin^2\theta +$$

$$+ \rho'(\theta)^2\sin^2\theta + \cancel{2\rho'(\theta)\rho(\theta)\cos\theta\sin\theta} + \rho(\theta)^2\cos^2\theta =$$

$$= \left(\rho(\theta)\right)^2 + \left(\rho'(\theta)\right)^2 \Rightarrow \ell(\gamma) = \int_a^b \sqrt{\rho^2 + (\rho')^2}\, d\theta$$

b) By 12 d (proved in class)

$$k(\theta) = \frac{|x'y'' - x''y'|}{(x'^2 + y'^2)^{3/2}} \rightarrow$$ (the absolute value has to be here!)

$x'' = \rho''\cos\theta - 2\rho'\sin\theta - \rho\cos\theta$ $\Rightarrow$

$y'' = \rho''\sin\theta + 2\rho'\cos\theta - \rho\sin\theta$

$$x''y' - x'y'' = \left(\rho''\cos\theta - 2\rho'\sin\theta - \rho\cos\theta\right)\left(\rho'\sin\theta + \rho\cos\theta\right) -$$

$$- \left(\rho''\sin\theta + 2\rho'\cos\theta - \rho\sin\theta\right)\left(\rho'\cos\theta - \rho\sin\theta\right) =$$

$$= \cancel{\rho''\rho'\cos\theta\sin\theta} - 2(\rho')^2\sin^2\theta - \cancel{\rho\rho'\cos\theta\sin\theta} + \rho\rho''\cos^2\theta - \cancel{2\rho'\rho\cos\theta\sin\theta} -$$

$$- \rho^2\cos^2\theta - \cancel{\rho''\rho'\sin\theta\cos\theta} - 2(\rho')^2\cos^2\theta + \cancel{\rho\rho'\sin\theta\cos\theta} +$$

$$+ \rho''\rho\sin^2\theta + \cancel{2\rho'\rho\sin\theta\cos\theta} - \rho^2\sin^2\theta =$$

$$= \rho''\rho - 2(\rho')^2 - \rho^2 \Rightarrow$$

$$k(t) = \frac{|\rho^2 + 2(\rho')^2 - \rho''\rho|}{(\rho^2 + (\rho')^2)^{3/2}}$$ - the absolute value has to be here!

(14) We can assume that α is parametrized by arc length

Let $f(t) = |\alpha(t)|^2 = \langle \alpha(t), \alpha(t) \rangle$

~~Since t_0 is the point of maximum of $f(t)$,~~

$f'(t_0) = 0$ and $f''(t_0) \leq 0$

$f'(t_0) = 2\langle \alpha'(t_0), \alpha(t_0) \rangle = 0 \Rightarrow \alpha(t_0) \perp \alpha'(t_0) \quad (*)$

$$f''(t_0) = 2\Big(\underbrace{\langle \alpha'(t_0), \alpha'(t_0) \rangle}_{1} + \langle \underbrace{\alpha''(t_0)}_{k(t_0)n(t_0)}, \alpha(t_0) \rangle\Big)$$

From $(*)$ $\alpha(t_0) = \pm |\alpha(t_0)| n(t_0)$ but

$\alpha(t_0)$ can't be equal to $|\alpha(t_0)| n(t_0)$

(otherwise $f''(t_0) = 2(1 + k(t_0)|\alpha(t_0)|) > 0$) $\Rightarrow$

$\alpha(t_0) = -|\alpha(t_0)| n(t_0) \Rightarrow$

$f''(t_0) = 2(1 - k(t_0)|\alpha(t_0)|) \leq 0 \Rightarrow$

$$k(t_0) \geq \frac{1}{|\alpha(t_0)|} \quad \text{q.e.d.}$$