Assignment 3 solutions, MATH 439

Section 1.5

10 a) The function e^{-1/t^2} is C^{∞}-function with all derivatives equal to 0 at $t=0$: It follows from the following 2 facts:

1) $\forall \alpha \in \mathbb{R}$ $\lim\limits_{t\to 0} \dfrac{e^{-1/t^2}}{t^\alpha} \underset{\text{substitute}}{=} \lim\limits_{x\to\infty} x^{\alpha} e^{-x^2} = 0$

2) The derivatives of any order at $t\neq 0$ have a form $P(\tfrac{1}{t})\, e^{-1/t^2}$ where P is a polynomial.

b) $\alpha'(t) = \big(1, 0, (e^{-1/t^2})'\big)$ (if $t \geq 0$) or $\big(1, (e^{-1/t^2})', 0\big)$ (if $t \leq 0$), and $\alpha'(t) \neq 0$ because the first component of $\alpha'(t)$ is a nonzero constant.

The curvature is $0 \Leftrightarrow \alpha''(t) \parallel \alpha'(t)$. If $t\neq 0$ $(e^{-1/t^2})' = \frac{2}{t^3} e^{-1/t^2}$

If $t \neq 0$ $\alpha''(t) = \left(0, 0, \left(-\frac{6}{t^4} + \frac{4}{t^6}\right) e^{-1/t^2}\right)$ or $\left(0, \left(-\frac{6}{t^4} + \frac{4}{t^6}\right) e^{-1/t^2}, 0\right)$

$\left(\frac{4}{t^6} - \frac{6}{t^4}\right) e^{-1/t^2} = 0 \Leftrightarrow 2 - 3t^2 = 0 \Leftrightarrow t = \pm\sqrt{\frac{2}{3}}$

If $t = 0$ $\alpha''(t) = (0,0,0) \Rightarrow k(0) = 0$ as well

c) If $t<0$ and $t \neq -\sqrt{\frac{2}{3}}$, then the osculating plane $=$ span $\langle \alpha'(t), \alpha''(t) \rangle$ = xy-plane $\Rightarrow$ its limit when $t \to 0-$ is xy-plane too

If $t>0$ and $t \neq \sqrt{\frac{2}{3}}$, then the osculating plane $=$ span $\langle \alpha'(t), \alpha''(t) \rangle$ = xz-plane $\Rightarrow$ its limit when $t\to 0+$ is xz-plane

d) Since both for $t<0$ and $t>0$ the corresponding piece of the curve lies in a plane then $\tau(t) = 0$ for $t \neq 0, \pm\sqrt{\frac{2}{3}}$

So it can be defined as 0 at $t=0$ by passing to the limit

2- 12 c) Let s be a natural parameter and $s = \varphi(t)$,

$\tilde{\alpha}(s) = \alpha(\varphi(s))$, $\tilde{k}(s) := k(\varphi(s))$, $\tilde{\tau}(s) := \tau(\varphi(s))$ Then by ex. 2 p 22

(proved in class)

$$\tilde{\tau}(s) = -\frac{(\tilde{\alpha}'(s) \times \tilde{\alpha}''(s)) \cdot \tilde{\alpha}^{(3)}(s)}{|\tilde{k}(s)|^2} \qquad (1)$$

By ex. 12 b (proved in class) in the original parameter

$$k(t) = \frac{|\alpha'(t) \times \alpha''(t)|}{|\alpha'(t)|^3} \qquad (2)$$

Let us transform (1) to the required formula using the chain rule and (2):

$$\tilde{\alpha}'(s) = \alpha'(\varphi(s))\,\varphi'(s)$$

$$\tilde{\alpha}''(s) = \alpha'(\varphi(s))\,\varphi''(s) + \alpha''(\varphi(s))(\varphi'(s))^2$$

$$\tilde{\alpha}^{(3)}(s) = \alpha'(\varphi(s))\,\varphi^{(3)}(s) + 3\alpha''(\varphi(s))\,\varphi'(s)\varphi''(s) + \alpha^{(3)}(\varphi(s))(\varphi'(s))^3$$

$$\Downarrow$$

$$(\tilde{\alpha}'(s) \times \tilde{\alpha}''(s)) \cdot \tilde{\alpha}^{(3)}(s) = (\alpha'(t) \times \alpha''(t)) \cdot \alpha^{(3)}(t) \cdot (\varphi'(s))^6 \qquad (3)$$

Rem As a matter of fact here it was important only that

$$\tilde{\alpha}''(s) = \alpha''(\varphi(s))(\varphi'(s))^2 + \dots \alpha'(\varphi(s)) \qquad (4)$$

$$\tilde{\alpha}^{(3)}(s) = \alpha^{(3)}(\varphi(s))(\varphi'(s))^3 + \dots \alpha''(\varphi(s)) + \dots \alpha'(\varphi(s)) \qquad (5)$$

The actual form of the coefficients in (4) and (5) denoted by ... is not important because they do not appear in the transformation of the scalar triple product

On the other hand by ex 12 a (proved in class)

$\varphi'(s) = \dfrac{1}{|\alpha'(t)|}$ (6) $\Rightarrow$ substitute (2), (3), (6) into (1)

$$\tau(t) = -\frac{(\alpha'(t) \times \alpha''(t)) \cdot \alpha^{(3)}(t) \cdot \cancel{|\alpha'(t)|^{-6}}}{|\alpha'(t) \times \alpha''(t)|^2} \cdot \cancel{|\alpha'(t)|^{6}}$$ q.e.d.

15 $|\tau(s)| = |b'(s)|$

$$n(s) = \frac{b'(s)}{\operatorname{sgn}\tau(s)\,|b'(s)|}$$

$$t(s) = n(s)\times b(s) = \frac{b'(s)\times b(s)}{\operatorname{sgn}\tau(s)\,|b'(s)|}$$

$$\Downarrow$$

$$k(s) = |t'(s)| = \left|\left(\frac{b'(s)\times b(s)}{|b'(s)|}\right)'\right|, \text{ i.e.}$$

$|\tau(s)|$ and $k(s)$ can be expressed in terms of $b(s)$ only.

17 a), b), c) (together) $\alpha'(s)$ make a constant angle with some vector $v \Leftrightarrow$

$\Leftrightarrow \underbrace{\langle \alpha'(s), v\rangle}_{\langle t(s), v\rangle} \equiv \text{const} \Leftrightarrow \underbrace{\langle \alpha''(s), v\rangle}_{kn} = 0 \Leftrightarrow \langle n(s), v\rangle = 0 \Leftrightarrow$ (This proves b))

$\langle b'(s), v\rangle = 0 \Leftrightarrow \langle b(s), v\rangle \equiv \text{const} \rightarrow$ (This proves c))

Besides, $\langle \underbrace{n'(s)}_{-k(s)t(s)-\tau(s)b(s)}, v\rangle = 0 \Rightarrow -k(s)\langle t(s), v\rangle - \tau(s)\langle b(s), v\rangle = 0 \Rightarrow$

$$\frac{k(s)}{\tau(s)} = -\frac{\langle b(s), v\rangle}{\langle t(s), v\rangle} \equiv \text{const}$$

So if $\langle \alpha'(s), v\rangle \equiv \text{const} \Rightarrow \frac{k(s)}{\tau(s)} \equiv \text{const}$ This proves one direction of a): $\langle \alpha'(s), v\rangle = \text{const} \Rightarrow \frac{k(s)}{\tau(s)}$ is a constant.

In an opposite direction if $-\frac{k}{\tau} = \text{const} = \operatorname{tg}\theta$ for some θ

~~take $v(s) = \cos\theta\, t(s)\ \sin\theta\, b(s)$~~

$$\text{Then } v'(s) = \cos\theta\, t'(s) + \sin\theta\, b'(s) = \underbrace{(\cos\theta\, k(s) + \sin\theta\, \tau(s))}_{0}\, n(s) = 0$$

$\Downarrow$

$v(s)$ is a constant vector and

$$\langle t(s), v\rangle = \cos\theta \equiv \text{const}$$

d) First s is a natural parameter:

$$\alpha'(s) = \left(\frac{a}{c}\sin\theta(s), \frac{a}{c}\cos\theta(s), \frac{b}{c}\right)$$ Let $v = (0,0,1)$

$$|\alpha'(s)| = \sqrt{\frac{a^2+b^2}{c^2}} = 1$$ Then $\langle \alpha'(s), v\rangle = \frac{b}{c} = \text{const} \overset{(a)}{\Rightarrow}$ α is a helix

$$\alpha''(s) = \left(\frac{a}{c}\cos\theta(s)\,\theta'(s), -\frac{a}{c}\sin\theta(s)\,\theta'(s), 0\right)$$

$$k(s) = |\alpha''(s)| = \frac{a}{c}|\theta'(s)| \quad (1)$$

$$\alpha^{(3)}(s) = \left(-\frac{a}{c}\sin\theta(s)(\theta'(s))^2, -\frac{a}{c}\cos\theta(s)(\theta'(s))^2, 0\right) +$$

$$+\left(\frac{a}{c}\cos\theta(s)\,\theta''(s), -\frac{a}{c}\sin\theta(s)\,\theta''(s), 0\right)$$

$$\alpha'(s)\times\alpha''(s) = \left(\frac{ab}{c^2}\sin\theta(s)\theta'(s), -\frac{ab}{c^2}\cos\theta(s)\theta'(s), -\frac{a^2}{c^2}\theta'(s)\right)$$

$$(\alpha'(s)\times\alpha''(s))\cdot\alpha^{(3)}(s) = -\frac{a^2 b}{c^3}(\theta'(s))^3$$

$$\tau(s) = -\frac{(\alpha'(s)\times\alpha''(s))\cdot\alpha^{(3)}(s)}{(k(s))^2} = \frac{\cancel{a^2} b}{\cancel{c^3}}\cdot\frac{\cancel{c^2}}{\cancel{a^2}}\cdot\frac{(\theta'(s))^3}{(\theta'(s))^2} = \frac{b}{c}\theta'(s) \quad (2)$$

(1) and (2) $\Rightarrow$

$$\boxed{\frac{k(s)}{\tau(s)} = \frac{a}{b}}$$

Section 1.6

1. Assume that P is a plane satisfying cond. 1 & 2 for $s = s_0$
Assume that v is a nonzero vector s.t. $v \perp P$
and let $f(s) = \langle v, \alpha(s)\rangle$
Then cond 1 $\Leftrightarrow$ $f'(s_0) = 0$ and cond 2 $\Leftrightarrow$ $f''(s_0) = 0$. In other words $\langle v, \alpha'(s_0)\rangle = \langle v, \alpha''(s_0)\rangle = 0 \Rightarrow$ $v \perp \text{span}(\alpha'(s_0), \alpha''(s_0)) \Rightarrow P = \text{span}(\alpha'(s_0), \alpha''(s_0)) =$ osculating plane