Homework Assignment 2 in Geometric Control Theory, MATH666, Fall 2013

due Oct 16, 2013

1. Let $M = SO(4)$, the group of all 4×4 orthogonal matrices with determinant equal to 1. Fix some <u>non-zero</u> number α and let:

$$A = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\alpha \\ 0 & 0 & \alpha & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Consider the following control system with the state space M:

$$\dot{E} = E(A + uB), \quad E \in M, \ u \in \{-1, 1\}. \tag{1}$$

where $E \in M$ and $u \in \{-1, 1\}$.

(a) Prove that system (1) is controllable if and only if $\alpha \neq \pm 1$.

Note that $\dim SO(4) = 6$ (in general $\dim SO(n) = \frac{n(n-1)}{2}$). When you proceed the calculations, instead of writing down matrices I recommend you to use the following notations: let E_{ij} be the 4×4 matrix such that its (i,j)th entry is equal to 1 and all other entries are equal to 0. For example, in this notation $B = E_{32} - E_{23}$ and $A = E_{21} - E_{12} + \alpha(E_{43} - E_{34})$. The following simple formula can be useful: $E_{ij}E_{kl} = \delta_{jk}E_{il}$, where δ_{jk} is the Kronecker index. In other words, $E_{ij}E_{kl} = 0$ if $j \neq k$ and it is equal to E_{il} if $j = k$.

(b) Will the answer of the previous item change if $u \in \{2, 3\}$ (instead of $\{-1, 1\}$)? Justify your answer.

(c) Assume that $\alpha = \pm 1$. Prove that for any point $E \in M$ the attainable set from E w.r.t. (1) coincides with the orbit of E w.r.t. (1) and find the dimension of every orbit.

(d) (**bonus-25 points**) Assume (for definiteness) that $\alpha = 1$. Let (e_1, e_2, e_3, e_4) be the standard basis in $\mathbb{R}^4$. Define the multiplication by the imaginary unit i on $\mathbb{R}^4$ by setting $ie_1 = -e_4$, $ie_4 = e_1$, $ie_2 = e_3$, $ie_3 = -e_2$. It defines the structure of two dimensional complex vector space on $\mathbb{R}^4$, $\mathbb{R}^4 \simeq \mathbb{C}^2$. Namely, the multiplication of a complex number to a vector is defined and any vector can be uniquely represented as a linear combination with complex coefficients of some two vectors (for example, of e_1 and e_2). Show that a matrix D belongs to the tangent space at the identity I to the orbit (of the identity) w.r.t (1) if and only if the corresponding linear operator $\widehat{D}$ is also linear over $\mathbb{C}$ (i.e $D(zv) = zD(v)$ for any $v \in \mathbb{R}^4$ and $z \in \mathbb{C}$) and the 2×2 complex matrix D_1 corresponding to this operator in the complex basis (e_1, e_2) satisfies: $D_1 = -\bar{D_1}^T$ (where $^-$ stands for the complex conjugation).

Remark: In other words, this item shows that in the case $\alpha = 1$ the orbit of the identity is the unitary group $U_2(\mathbb{C}) = U_4(\mathbb{R})$. Similar conclusion (with slightly modified complex structure on $\mathbb{R}^4$) can be done for $\alpha = -1$.