Extra Credit assignment on Grassmannians Spring 2016 - MATH622

(equal to a half of a regular homework assignment)

**due February 11, 2016 at the beginning of class**

The text below contains some definitions, constructions, remarks, and two problems. The formulations of the problems are given in an oblique font and are ended by $\Box$. You are asked to submit the solutions to these two problems only.

Given a positive integers $k$, $n$ with $k \leq n$ the *Grassmannian* $\mathrm{Gr}(k,n)$ is the set of $k$-dimensional subspaces of $\mathbb{R}^n$. Further, given an $(n-k)$-dimensional subspace $\Lambda \in \mathbb{R}^n$ let $\Lambda^{\pitchfork}$ be the set of all $k$-dimensional planes that are transversal to $\Lambda$,

$$\Lambda^{\pitchfork} = \{L \in \mathrm{Gr}(k,n) : L + \Lambda = \mathbb{R}^n\}$$

(equivalently, $\Lambda^{\pitchfork} = \{L \in \mathrm{Gr}(k,n) : L \cap \Lambda = 0\}$)

**Problem 1** *Fix $L_0 \in \Lambda^{\pitchfork}$. Given any $L \in \Lambda^{\pitchfork}$ there exists a unique linear map $A_L : L_0 \to \Lambda$ such that $L = \{v + A_L(v) : v \in L_0\}$ (in other words, $L$ is the graph of the map $A_L$)* $\Box$

Note that vice versa given a linear map $A : L_0 \mapsto \Lambda$ there exist a unique $L \in \Lambda^{\pitchfork}$ such that $A = A_L$.

Further, denote by $\mathcal{M}_{(n-k)\times k}$ the space of $(n-k) \times k$ matrices. Fixing a basis $E = (e_1, \ldots, e_k)$ in $L_0$ and $F = (f_1, ..., f_{n-k})$ in $\Lambda$, let $S_L$ be the $(n-k) \times k$-matrix representing the operator $A_L$ in the bases $E$ and $F$. This defines the bijective map $\Psi_{E,F} : \Lambda^{\pitchfork} \to \mathcal{M}_{k\times(n-k)}$ such that $\Psi_{E,F}(L) = S_L$. Keeping in mind that $\mathcal{M}_{k\times(n-k)}$ can be identified with $\mathbb{R}^{k(n-k)}$, we obtain a chart $(\Lambda^{\pitchfork}, \Psi_{E,F})$ with $\Psi_{E,F} = \mathcal{M}_{(n-k)\times k} \cong \mathbb{R}^{k(n-k)}$.

Now take another $(n-k)$-dimensional subspace $\widetilde{\Lambda}$ and $\widetilde{L}_0 \in \widetilde{\Lambda}^{\pitchfork}$. Choose a basis $\widetilde{E} = (\tilde{e}_1, \ldots, \tilde{e}_k)$ in $\widetilde{L}_0$ and $\widetilde{F} = (\tilde{f}_1, \ldots, \tilde{f}_{n-k})$ in $\widetilde{\Lambda}$. Then, similarly to the previous paragraph, one can define the chart $(\widetilde{\Lambda}^{\pitchfork}, \Psi_{\widetilde{E},\widetilde{F}})$ based on this new data.

Note that both tuples of $n$ vectors $(E, F)$ and $(\widetilde{E}, \widetilde{F})$ form bases in $\mathbb{R}^n$. Therefore there exist matrices $A$, $B$, $C$, $D$ of sizes $k \times k$, $(n-k) \times k$, $k \times (n-k)$, and $(n-k) \times (n-k)$ respectively such that

$$E = \widetilde{E}A + \widetilde{F}B, \quad F = \widetilde{E}C + \widetilde{F}D$$

(here $\widetilde{E}A$ means the multiplication of the row of vectors $(\tilde{e}_1, \ldots, \tilde{e}_k)$ on the matrix $A$ etc).

**Problem 2** *Express the transition map $\Phi_{\widetilde{E},\widetilde{F}} \circ (\Phi_{E,F})^{-1}$ between the charts in terms of the matrices $A$, $B$, $C$, and $D$ and prove that it is $C^\infty$ (even $C^\omega$) smooth on the domain of its definition (i.e. on $\Phi_{E,F}(\Lambda^{\pitchfork} \cap \widetilde{\Lambda}^{\pitchfork})$)*$\Box$.

Problem 2 will imply that the collection of charts of the form $(\widetilde{\Lambda}^{\pitchfork}, \Psi_{\widetilde{E},\widetilde{F}})$ defines the $C^\infty$ ($C^\omega$) differential structure on $\mathrm{Gr}(k,n)$, where the basis for the topology on $\mathrm{Gr}(k,n)$ is given by the preimages of open sets of $\mathcal{M}_{(n-k)\times k} \cong \mathbb{R}^{k(n-k)}$ under the maps $\Psi_{\widetilde{E},\widetilde{F}}$.

**Remark** An equivalent way to define the differential structure on $\mathrm{Gr}(k,n)$ is by looking on it as on the homogeneous space of the general linear group $GL_n$ over the subgroup preserving some $k$- dimensional subspace in $\mathbb{R}^n$. Homogeneous spaces are discussed in Warner in chapter 3 starting from page 120 and in particular the very first theorem there describes the canonical way to introduce the differential structure on them.