# Geometry of rank 2 distributions via abnormal extremals: generalized Wilczynski invariants

Igor Zelenko

Texas A&M University, USA

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# Vector distributions: weak derived flag

Let $D$ be a rank $\ell$ distribution on an $n$ dimensional manifold $M$ or shortly $(\ell, n)$-distribution.

The natural filtration of $TM$, the weak derived flag:

$$D = D^1 \subset D^2 \subset \dots D^j \subset \dots \subset TM :$$

$$D^1(q) := D(q) = \langle X_1(q), \dots, X_l(q) \rangle,$$

$$D^2(q) := D(q) + [D, D](q) = \langle \{X_i(q), [X_i, X_k](q) : 1 \leq i < k \leq \ell\} \rangle,$$

and recursively

$$D^j(q) = D^{j-1}(q) + [D, D^{j-1}](q) =$$

$$= \mathrm{span}\,\{\text{all iterated Lie brackets of length} \leq j \text{ of the fields } X_i \text{at } q\}.$$

# Weak derived flag and small growth vector (continued)

$D^j$ is called the *$j$th power of the distributions $D$*

The filtration $D(q) = D^1(q) \subset D^2(q) \subset \ldots D^j(q), \ldots$ of the tangent bundle $T_qM$ is called a *weak derived flag*

The tuple $(\dim D(q), \dim D^2(q), \ldots, \dim D^j(q), \ldots)$ is called the *small growth vector of $D$ at the point $q$* (or, shortly, s.g.v.).

# Main approaches to the equivalence problem

1. The Cartan equivalence method.
2. The Tanaka Prolongation procedure -the algebraic version of the Cartan equivalence method (Tanaka 1970, Morimoto 1993) working especially well in parabolic geometries (Tanaka 1979, Čap-Schichl (2000), Čap-Slovak), as was discussed in Dennis The lecture series.
3. The method of normal forms (Poincare-Dulac for vector fields, Moser for stable distribution and nondegenerate CR structures (and many others for CR structures),Misha Zhitomirskii for distributions), as was discussd in Misha Zhitomirskii lecture series.
4. The symplecitification procedure via abnormal extremals and Jacobi curves (A. Agrachev, I.Z, and B. Doubrov) originated from the ideas of optimal control theory.

# Preliminaries on cotangent bundle: the tautological Liouville 1-form and the canonical symplectic structure

Let $T^*M = \{(p,q) : q \in M, p \in T_q^*M\}$ be the cotangent bundle, $\pi : T^*M \to M$ be the canonical projection.

$T^*M$ $\quad\quad$ $\lambda = (p,q) \in T^*M, \mathrm{v} \in T_\lambda T^*M$

$\pi \downarrow$ $\quad\quad$ $q \in M, \quad p \in T_q^*M$ $\quad\quad$ $\downarrow \pi_*$

$M$ $\quad\quad$ $\pi_* v \in T_q M$

The tautological Liouville 1-form $s$ on $T^*M$ is $s(\lambda)(v) := p(\pi_* v)$
The canonical symplectic form on $T^*M$ is $\sigma := d\,s$.
In local (canonical) coordinates $s = p_i dq^i$ and $\sigma = dp_i \wedge dq^i$

# The projectivized cotangent bundle

Let $\mathbb{P}T^*M$ be the projectivized cotangent bundle: the fibers are the projectivizations of the fibers of $T^*M$.

The tautological $1$-form $s$ induces the canonical contact distribution $\widetilde{\Delta}$ on $\mathbb{P}T^*M$ as a pushforward of the distribution annihilating $s$ by the projection from $T^*M$ to $\mathbb{P}T^*M$:

$$\begin{array}{ccc} T^*M & \qquad & \ker s \\ \Pi \downarrow & & \downarrow \Pi_* \\ \mathbb{P}T^*M & & \widetilde{\Delta} = \Pi_* \ker s \end{array}$$

# Annihilators of powers of distributions and structures on them

Dual objects to the powers of distributions on $T^*M$ and $\mathbb{P}T^*M$:

$(D^j)^\perp = \{(p,q) \in T^*M : p(v) = 0 \quad \forall v \in D^j(q)\}$ - the annihilator of $D^j$

$\mathbb{P}(D^j)^\perp$ is the projectivization of $(D^j)^\perp$.

Consider the case of rank $2$ distributions with $\dim D^2 = 3$.

Note that $\dim \mathbb{P}(D^2)^\perp = 2n - \dim D^2 - 1 = 2n - 4$ ($\Rightarrow \dim \mathbb{P}(D^2)^\perp$ it is even).

Restrict the canonical contact distribution $\widetilde{\Delta}$ from $\mathbb{P}T^*M$ to $\mathbb{P}(D^2)^\perp$:

$$\bar{\Delta} := \widetilde{\Delta} \cap T\mathbb{P}(D^2)^\perp$$

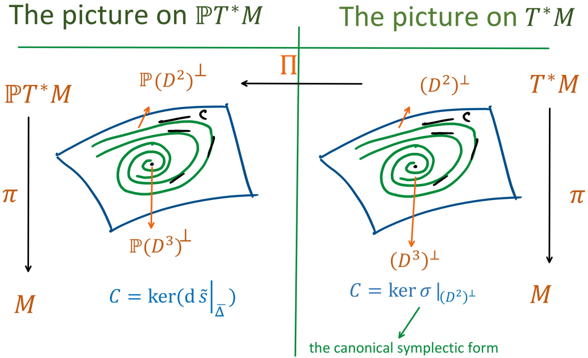
The distribution $\bar{\Delta}$ is even contact on $\mathbb{P}(D^2)^\perp \backslash \mathbb{P}(D^3)^\perp$, i.e if $\tilde{s}$ is a defining $1$-form of $\bar{\Delta}$, $\bar{\Delta} = \ker \tilde{s}$, then on $\mathbb{P}(D^2)^\perp \backslash \mathbb{P}(D^3)^\perp$

$$\dim \ker(d\tilde{s}|_{\bar{\Delta}}) = 1.$$

# Characteristic foliation (by abnormal extremals)

$\mathcal{C} := \ker(d\tilde{s}|_{\bar{\Delta}})$ is the the *characteristic rank 1 distribution* on $W_D = \mathbb{P}(D^2)^\perp \setminus \mathbb{P}(D^3)^\perp$.

The integral curves of this characteristic distribution are *(regular)* abnormal extremals of distribution $D$, defining the *characteristic 1-foliation on $\mathcal{W}_D$*.



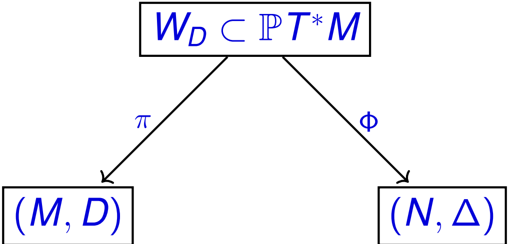
# Leaf space of abnormal extremals and the double fibration

$$N = W_D/(\text{the charactrestic one-foliation of abnormal extremals})$$

is locally a well defined smooth $(2n-5)$-dimensional manifold, the *leaf space of abnormal extremals*.

Let $\Phi : W_D \to N$ be the canonical projection to the quotient manifold.

The leaf space $N$ is endowed with the contact distribution $\Delta := \Phi_* \bar{\Delta}$, , $\mathrm{rank}\Delta = 2n-6$, $\Delta$ is endowed with the conformal symplectic structure.



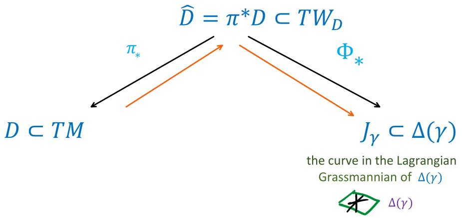
# Rank 2 distribution of maximal class and curve in projective spaces

Let $\widehat{D} := \pi^* D$, be the distribution on $W_D$ induced by $\pi$:

$$\widehat{D}(\lambda) = \{v \in T_\lambda W_D : \pi_* v \in D(\pi(\lambda))\}$$

$$\forall \lambda \in \gamma \quad J_\gamma(\lambda) := \Phi_*(\widehat{D}(\lambda)) \subset \Delta(\gamma)$$

$J_\gamma$ is an (unparametrized) curve of (Langrangian) subspaces of $\Delta(\gamma) \subset T_\gamma N$, called the Jacobi curve of the abnormal extremal $\gamma$.




**Remark** $\forall q \in M$ collecting $\pi_* \mathcal{C}$ along the fiber $\pi^{-1}(q)$ of $\pi : W_D \to M$ we do not get any non-trivial structure on $D(q)$.

# Osculating flag of Jacobi curve

The Jacobi curve $J_\gamma$ produces the curve of flags in $\Delta(\gamma)$ via a series of osculations and skew-orthogonal complements:

$$\cdots \subset J_\gamma^{(-\nu)} \subset \cdots \subset J_\gamma^{(0)} = J_\gamma \subset J_\gamma^{(1)} \subset \cdots \subset J_\gamma^{(\nu)} \subset \cdots \subset \Delta(\gamma)$$

Here

1. $J_\gamma^{(i)}$ with $i \geq 0$ is the $i$-th osculating space defined as follows: Look on $J_\gamma(\cdot)$ as a tautological vector bundle over itself with the fiber over the point $J_\gamma(t)$ being vector space $J_\gamma(t)$. Let $\Gamma(J_\gamma)$ be the space of sections of this bundle, then $J_\gamma^{(i)}(t) = \mathrm{span}\{\frac{d^j}{d\tau^j}\ell(\tau)|_{\tau=t} : \ell \in \Gamma(J_\gamma), 0 \leq j \leq i\}$.
2. $J_\gamma^{(-i)} := \left(J_\gamma^{(i)}\right)^\angle$, the skew-symmetric complement of $J_\gamma^{(i)}$.

For rank $2$ distributions, $\dim J_\gamma^{(i+1)} - \dim J_\gamma^{(i)} \leq 1$.
$J_\gamma^{(-1)}(\lambda) = \Phi_*(\mathcal{V}(\lambda))$, $\lambda \in \gamma$, where $\mathcal{V}$ is the distribution tangent to the fibers of the bundle $\pi : W_D \to M$.

# Associated curves in projective space and distributions of maximal class

The curve $J_\gamma$ is called regular if the subspaces $J_\gamma(\lambda)$ do not belong to a fixed hyperplane of $\Delta(\gamma)$ $\Leftrightarrow$ For generic $\lambda \in \gamma$ the following three mutually equivalent conditions hold (in this case the curve is called convex):

1. $J_\gamma^{(n-3)}(\lambda) = \Delta(\gamma)$;
2. $\dim J_\gamma^{(i)} = i + n - 3$ for $3 - n \le i \le n - 3$;
3. $\dim J_\gamma^{(4-n)} = 1$, i.e. near $\lambda$, $\bar\lambda \mapsto J_\gamma^{(4-n)}(\bar\lambda)$, $\bar\lambda \in \gamma$, is the curve in the projective space $\mathbb{P}\Delta(\gamma)$ (moreover, it is the self-dual curve in the projective space)

Let $\mathcal{R}_D \subset W_D$, the Jacobi regularity locus of $D$, be the set of $\lambda \in W_D$ such that the germ of $J_\gamma(\lambda)$ at $\lambda$ is convex, where $\gamma$ is the abnormal extremal passing through $\lambda$.
The rank $2$ distribution $D$ is of maximal class at the point $q$ if $\mathcal{R}_D \cap \pi^{-1}(q)$ is not empty.
Therefore invariants of (self-dual) curves in projective spaces give invariants of rank $2$ distribution of maximal class in

# Remarks on distributions of maximal class

- Generic germs of rank $2$ distributions are of maximal class.

- No example of rank $2$ bracket generating distribution with $\dim D^3 = 5$, which are not of maximal class are known.

- A distribution $D$ is of maximal class at a given point , if the flat distribution corresponding to the Tanaka symbol of $D$ at $q$ is of maximal class.

- With Eric Wendel we have shown that the following 3 classes of bracket generating distributions with $\dim D^3 = 5$ are of maximal class:
  1. degree of nonholonomy $\leq 4$;
  2. $(2, 14)$-distribution with free small growth vector $(2, 3, 5, 8, 14)$;
  3. if a distribution is associated with a Monge equation $y^{(m)} = F(x, y, y', \dots, y^{(m-1)}, z, z', \dots, z^{(k)})$, $m + k \geq 3$, $F_{z^{(k)} z^{(k)}} \neq 0$.

# Geometry of curves in projective space: main points

1. *Canonical projective structure on a curve*: i.e. the set of distinguished parametrizations defined up to a Möbius transformation.
2. If $k$ is the dimension of the projective space, then for a convex curve in the projective space the set of fundamental invariants consists of $k-1$ relative invariants $\mathcal{W}_i$ of degree $i+2$, $i=1,\dots,k-1$, called the Wilczynski invariants. Here $\mathcal{W}_i$ is a degree $i+2$ homogeneous polynomial on the tangent line at every point of the curve. In the given parametrization $t$ it can be written as $\mathcal{W}_i(t)=A_i(t)\,dt^{i+2}$. The function $A_i(t)$ is called the density of the Wilczynski invariant w.r.t. the parameter $t$.
3. The curve in a projective space is self-dual if and only if all Wilczynski invariants of odd degree are equal to zero.

# Canonical section of parametrized curve

First assume that the curve $J$ in a $k$ dimensional projective space $\mathbb{P}V$ of a vector space $V$ is parametrized somehow: $t \mapsto J(t)$.
Let $t \mapsto E(t)$ be a section of $J$ (considered as the tautological bundle over itself).
The convexity assumption is that $E(t), E'(t), \ldots, E^{(k)}(t)$ constitute a basis of $V$.
Among all sections of $J$ (the freedom is $E(t) \mapsto \lambda(t)E(t)$ for a nonzero scalar function $\lambda(t)$) there is the unique section, up to a multiplication by a constant, such that

$$\frac{d^{k+1}}{dt^{k+1}}E(t) = \sum_{i=0}^{k-1} B_i(t)\frac{d^i}{dt^i}E(t),$$

called the canonical section of $J$ (i.e. $B_k \equiv 0$) w.r.t. to the chosen parametrization.
**Explanation:** $B_k \to B_k + (k+1)\frac{\lambda'}{\lambda} \Rightarrow B_k \to 0 \Leftrightarrow \lambda' = -\frac{1}{k+1}B_k\lambda$.

# Canonical projective structure (continued)

Among all parametrizations of $J$ there are parametrizations such that (for their canonical sections):

$$\frac{d^{k+1}}{dt^{k+1}}E(t) = \sum_{i=0}^{k-2} B_i(t)\frac{d^i}{dt^i}E(t),$$

i.e. $B_k = B_{k-1} = 0$ -the Laguerre -Forsyth canonical form.

Such parametrizations are defined up to a Möbius transformation and called *projective parameters*.
The collection of them define the *canonical projective structure on the curve $J$*.

**Explanation**: Under reparametrization $\tau = \varphi(t)$,
$B_{k-1}(t) \to (B_{k-1} + c_k \mathbb{S}(\varphi t)) \left(\frac{dt}{d\tau}\right)^2$, where $\mathbb{S}(\varphi) := \frac{\varphi^{(3)}}{\phi'} - \frac{3}{2}\left(\frac{\varphi''}{\phi'}\right)^2$ is the Schwarzian derivative of $\varphi$ and $c_k = \frac{k+1)(k+2)}{12} \Rightarrow$
$B_{k-1} \to 0 \Leftrightarrow \mathbb{S}(\varphi) = -(c_k)^{-1} B_{k-1}$.

# The Wilczynski invariants

Now assume that $t$ is a projective parameter on $J$.

$$\frac{d^{k+1}}{dt^{k+1}}E(t) = \sum_{i=0}^{k-2} B_i(t)\frac{d^i}{dt^i}E(t), \qquad (1)$$

Then the form $\mathcal{W}_1 = B_{k-2}(dt)^3$ is independent of the choice of the projective parameter-the *Wilczynski invariant of degree* 3, i.e if $\tau$ is is another projective parameter and the coefficient $\widetilde{B}_{k-1}(\tau)$ is as in the decomposition (1) , then

$$\widetilde{B}_{k-2}\,(d\tau)^3 = B_{k-2}(dt)^3.$$

More generally, the degree $i+2$ relative invariant

$$\mathcal{W}_i(t) \overset{def}{=} \frac{(i+1)!}{(2i+2)!}\left(\sum_{j=1}^{i} \frac{(-1)^{j-1}(2i-j+3)!(k-i+j-2)!}{(i+2-j)!(j-1)!} B^{(j-1)}_{k-2-i+j}(t)\right)(dt)^{i+2}$$

on $J$ does not depend of the choice of the projective parameter-the $i$th Wilczynski invariant, $1 \le i \le k-1$. (an alternative description using $\mathfrak{sl}_2$-representations -Y. Se-Ashi (1988) , B. Doubrov (2007))

# Wilczynski invariants of self-dual curves

Given a convex curve $J$ in $\mathbb{P}V$ the dual curve $J^*$ in $\mathbb{P}V^*$ consist of lines in $\mathbb{P}V^*$ annihilating the hyperplanes $J^{(k-1)}$ obtained from $J$ by the osculation of order $k-1$.
The curve $J$ is called *self-dual* if it is equivalent to its dual, i.e. there is a linear transformation $A : V \mapsto V^*$ sending $J$ onto $J^*$.

If $k = 2m - 1$ then $J$ is self-dual if an only if there exists a symplectic form $\omega$ on $V$ such that the curve $J^{(m-1)}$ of $(m-1)$st osculating subspaces of $J$ is Lagrangian w.r.t. $\omega$.

**Theorem (Wilczynski, 1905)** The curve is self-dual if and only if all Wilczynski invariants of odd degree vanish.

In particular, the first nontrivial Wilczynski invariant is of degree 4: $\mathcal{W}_2 = B_{k-3}(t)dt^4$.

For Jacobi curves of $(2,5)$-distributions $k = 3$ and $\mathcal{W}_2$ is the only nontrivial Wilczynski invariant.

# From curves in projective spaces back to distributions: Hamiltonian formalism

On the level of distribution: Let $D = \text{span}\{X_1, X_2\}$-local basis

$$X_3 := [X_1, X_2], \quad X_4 := [X_1, X_3], \quad X_5 := [X_2, X_3].$$

- Let us introduce the "quasi-impulses" of the vector fields $X_i$, $u_i : T^*M \mapsto \mathbb{R}$, $1 \leq i \leq 5$.

$$u_i(\lambda) := p \cdot X_i(q), \ \lambda = (p, q), \ q \in M, \ p \in T^*_q M.$$

  Then $(D^2)^\perp = \{\lambda \in T^*M : u_1(\lambda) = u_2(\lambda) = u_3(\lambda) = 0\}$.
- To any function $H : T^*M \mapsto \mathbb{R}$ corresponds the *Hamiltonian vector field* $\overrightarrow{H}$ defined by the relation

$$i_{\overrightarrow{H}}\sigma = -d\,H$$

  Then the characteristic rank 1 distribution $\mathcal{C}$ on $W_D$ satisfies $\mathcal{C} = \langle u_4 \overrightarrow{u}_2 - u_5 \overrightarrow{u}_1 \rangle$.

# Density with respect to local basis

Let $\vec{h}_{X_1,X_2} := u_4 \overrightarrow{u}_2 - u_5 \overrightarrow{u}_1$.

Let $\mathcal{R}_D \subset W_D$ be the Jacobi regularity locus of $D$ , i.e. the set of $\lambda \in W_D$ such that the germ of $J_\gamma(\lambda)$ at $\lambda$ is convex, where $\gamma$ is the abnormal extremal passing through $\lambda$.

For any $\lambda \in \mathcal{R}_D$, let $\mathcal{W}_{2i}^\lambda$ be the $2i$th Wilczynski invariants of the Jacobi curve $J_\gamma$ at $\lambda$, $1 \leq i \leq n-4$.

$\mathcal{W}_{2i}^\lambda$ is a degree $2(i+1)$ homogeneous function on the tangent line to $\gamma$ at $\lambda$.
To any (local) basis $(X_1, X_2)$ of $D$ we assign the following real-valued function on $\mathcal{R}_D$

$$A_i^{X_1,X_2}(\lambda) := \mathcal{W}_{2i}^\lambda(\vec{h}_{X_1,X_2}(\lambda)).$$

If $t \mapsto J_\gamma(e^{t\vec{h}_{X_1,X_2}}\lambda)$ is a parametrization of $J_\gamma$, then $A_i^{X_1,X_2}(\lambda)$ is the density of the $i$th Wilcynski invariant of this curve w.r.t. the parametrization $t$ at $t = 0$.

# The effect of a basis change and generalized Wilczynski invariants of rank 2 distributions

If $(\widetilde{X}_1, \widetilde{X}_2)$ is another basis of the distribution $D$, $(\widetilde{X}_1, \widetilde{X}_2) = (X_1, X_2)U$, $U \in GL_2(\mathbb{R})$, then

$$\vec{h}_{\widetilde{X}_1, \widetilde{X}_2}(\lambda) = \det U\big(\pi(\lambda)\big)^2 \big(\pi(\lambda)\big)\vec{h}_{X_1, X_2}(\lambda)$$

$\Downarrow$ homogeneity of $\mathcal{W}_{2i}$

$$A_i^{\widetilde{X}_1, \widetilde{X}_2}(\lambda) = \det U\big(\pi(\lambda)\big)^{4(i+1)} A_i^{X_1, X_2}(\lambda) \qquad (2)$$

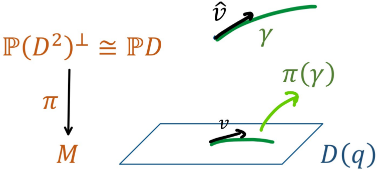
i.e., the restriction $A_i^{X_1, X_2}$ to $\mathcal{R}_D(q) := \mathcal{R}_D \cap \pi^{-1}(q)$ is the well defined function, up to the multiplication on a positive constant, the $i$th generalized Wilczynski of $D$

In the sequel we will use $\vec{h} := \vec{h}_{X_1, X_2}, \quad A_i := A_i^{X_1, X_2}$.

$A_i|_{\mathcal{R}_D(q)}$ is a degree $2(i+1)$ homogeneous rational function on $(D^2)^\perp(q)$.

# Tangential generalized Wilczynski invariant of (2,3,5)-distribution and the Cartan quartic.

- $\mathcal{R}_D = W_D$ ($= \mathbb{P}(D^2)^\perp$, as $\mathbb{P}(D^3)^\perp = \emptyset$ ), i.e. all Jacobi curves are convex;
- $\forall v \in D(q)$ there exist a unique $\lambda \in \mathbb{P}(D^2)^\perp \cap \pi^{-1}(q)$ and the unique $\widehat{v} \in \mathcal{C}(\lambda)$ such that $\pi_* \widehat{v} = v$




- The map $v \mapsto \mathcal{W}_2^\lambda(\widehat{v})$ is well defined degree 4 homogeneous function on $D(q)$, called the tangential generalize Wilczynski invariant of (2,5) distribution, and denoted by $\mathcal{W}_2$.

**Theorem**

*(I. Z.-2006)* $\mathcal{W}_2 = -$ *Cartan's quartic.*

# How to calculate the Wilczynski invariants?

First, on the level of a curve $J$ in Lagrangian Grassmannian $LG(V)$ of a $2m$-dim. symplectic space $(V, \omega)$ (keep in mind that for $(2, n)$-distributions $m = n - 3$):

Step 1 Find the osculating flag, in particular, check whether $J^{(1-m)}$ is one-dimensional.

Step 2 Choose some section $\widetilde{E}$ of $J^{(1-m)}$ and a parametrization (not necessary projective) of $J$ (by a parameter $t$). "Normalize" it to make it canonical w.r.t. to the chosen parameter. In fact, if $\alpha = \left|\omega\left(\widetilde{E}^{(m)}, \widetilde{E}^{(m-1)}\right)\right|$, then $E := \alpha^{-1/2}\widetilde{E}$ is a canonical section w.r.t. $t$.

Step 3 Instead of going further with Lagerre-Forsyth normalization to find a projective parameter, use universal polynomial formulas for the densities of Wilczynski invariants in terms of polynomials in the coefficients $\{B_i\}_{i=0}^{2m-2}$ and their derivatives w.r.t. this originally chosen parameter.

# Some formula for the first two nontrivial Wilczinski invariants of self-dual curves

If $\frac{d^{2m}}{dt^{2m}}E(t) = \sum_{i=0}^{2m-2} B_i(t)\frac{d^i}{dt^i}E(t)$, and $W_{2i}(t) = A_i(t)dt^{2(i+1)}$,
then
$A_1 = (2m-2)!\left(\frac{1}{(2m-2)(2m-3)}B_{2m-4} + \frac{(10m+7)}{20(4m^2-1)m}B_{2m-2}^2 - \frac{3}{20}B_{2m-2}''\right)$.

In particular:

- $m = 2$ (the case of $(2,5)$ distributions)
$$A_1 = B_0 + \frac{9}{100}(B_2)^2 - \frac{3}{10}B_2''$$
- $m = 3$ (the case of $(2,6)$ distributions)
$$A_1 = 2\left(B_2 + \frac{37}{175}(B_4)^2 - \frac{9}{5}B_4''\right)$$

If $m = 3$
$$A_2 = B_0 + \frac{1}{441}B_2B_4 + \frac{178}{15435}(B_4)^3 - \frac{5}{18}B_2'' - \frac{5}{441}(B_4')^2 - \frac{59}{441}B_4B_4'' + \frac{37}{7}B_4^{(4)}$$

On the level of a $(2, n)$-distribution $D$: Choose a local basis $(X_1, X_2)$ of $D$ again and let $\vec{h} = u_4 \overrightarrow{u}_2 - u_5 \overrightarrow{u}_1$.

$\vec{h}$ defines the parametrization on any abnormal extremal $\gamma$ and therefore on the Jacobi curve $J_\gamma$.

The operation $\frac{d}{dt}$ on sections of $J$ translates to the operation $\mathrm{ad}\, \vec{h}$ on appropriate vector fields on $\mathbb{P}(D^2)^\perp$ (or $(D^2)^\perp$).

**Step 1** For every $\lambda \in (D^2)^\perp$ find the osculating flag of $J_\gamma$ at $\lambda$, in particular, check whether $J^{(4-n)}$ is one-dimensional (in this way you also find the Jacobi regularity set $\mathcal{R}_D$).

**Step 2** Choose some section $\widetilde{\mathcal{E}}$ of the line distribution $J^{(4-n)}$. "Normalize" it to make it canonical w.r.t. to the parametrization given by $\vec{h}$: if $\alpha = \left|\sigma\left((\mathrm{ad}\, \vec{h})^m \widetilde{\mathcal{E}}, (\mathrm{ad}\, \vec{h})^{m-1} \widetilde{\mathcal{E}}\right)\right|$, then $\mathcal{E} := \alpha^{-1/2} \widetilde{\mathcal{E}}(t)$ is a canonical section $J^{(4-n)}$ w.r.t. the parametrization by $\vec{h}$

**Step 3** Find the decomposition of $(\mathrm{ad}\, \vec{h})^{2m} \mathcal{E}$ in the linear combination w.r.t. $\{(\mathrm{ad}\, \vec{h})^i \mathcal{E}\}_{i=0}^{2m-2}$ and use the universal polynomial formulas for the density of Wilczynski invariants in terms of universal polynomials in these coefficients and their directional derivative w.r.t. $\vec{h}$.

Suppose the canonical section $\mathcal{E}$ w.r.t. the parametrization by $\vec{h}$ is found, $m := n - 3$, and

$$(\mathrm{ad}\vec{h})^{2m}\mathcal{E} = \sum_{i=0}^{2m-2} \mathcal{B}_i(\mathrm{ad}\,\vec{h})^i\mathcal{E} \mod (\vec{h}, \text{Euler field})\rangle$$

Then the first generalized Wilczynski invariant is given by
$A_1 = (2m-2)!\left(\frac{1}{(2m-2)(2m-3)}\mathcal{B}_{2m-4} + \frac{(10m+7)}{20(4m^2-1)m}\mathcal{B}_{2m-2}^2 - \frac{3}{20}(\mathrm{ad}\,\vec{h})^2(\mathcal{B}_{2m-2})\right)$.
In particular:

- $m = 2$ (the case of $(2,5)$ distributions)
$$A_1 = \mathcal{B}_0 + \frac{9}{100}(\mathcal{B}_2)^2 - \frac{3}{10}(\mathrm{ad}\,\vec{h})^2\mathcal{B}_2(t)$$
- $m = 3$ (the case of $(2,6)$ distributions)
$$A_1 = 2\left(\mathcal{B}_2 + \frac{37}{175}(\mathcal{B}_4)^2 - \frac{9}{5}(\mathrm{ad}\,\vec{h})^2\mathcal{B}_4\right)$$
$A_2 :=$ the last formula on slide 24 with $\frac{d}{dt}$ replaced by $\mathrm{ad}\,\vec{h}$.

The natural (and widely open) questions are:

- How to describe the singularities of the vector field $\mathcal{E}$, or equivalently the set $S_D = W_D \backslash \mathcal{R}_D$, Jacobi singularity locus?
- how does this set depend on the Tanaka symbol of the distribution?
- What is the algebraic structure of generalized Wilczynski invariant and what more simple invariants can be extracted from it?

# Discussions on algebraic structure of generalized Wilczynski invariants

For $n = 5$, $J^{(4-n)} = \mathcal{V}$, the tangent to the fibers of the bundle $\pi : \mathbb{P}(D^2)^\perp \to D$.
$S_D = \emptyset$, $\mathcal{E}(\lambda) = \gamma_4(\lambda)\partial_{u_4} + \gamma_5(\lambda)\partial_{u_5}$, where $\gamma_4(\lambda)u_5 - \gamma_5(\lambda)u_4 \equiv 1$ ( e.g., $\mathcal{E} = \frac{1}{u_5}\partial_{u_4}$ or $-\frac{1}{u_4}\partial_{u_5}$ ).
Then the only generalized Wilczynski invariant $A_1$ is a degree $4$ polynomial on the fibers and can be computed using the formula in the previous slide.

For $n > 5$ the Jacobi singularity set $S_D$ is not empty and the generalized Wilczynski invariant are not polynomials for generic distributions (they are homogeneous rational functions). This is the case when (complexified) $S_D$ is not empty and the (complexified) characteristic line distribution $C$ is not tangent to the maximal strata of $S_D$.

# The case of (2,6) distribution with s.g.v. (2,3,5,6)

Each fiber of $D$ is endowed with a conformal structure given by

$$B(X,Y) := [X,[Z,Y]] \bmod D^3, \quad X,Y \in D, Z \in D^2/D$$

(note that $\dim D^2/D = 1$).
$B(X,Y) = B(Y,X)$ by Jacobi identity.
The Tanaka symbol of $D$ at $q$ is determined by the signature of $B$ and there are exactly three Tanaka symbols: elliptic ($B$ is sign definite), parabolic ($B$ is degenerate, of rank 1), and hyperbolic ($B$ has signature $(1,1)$):
One can choose a basis $(X_1,\ldots,X_6)$ in Tanaka symbols such that $\mathfrak{g}^{-1} = \langle X_1, X_2\rangle$, $X_3 = [X_1,X_2]$, $X_4 = [X_1,X_3]$, $X_5 = [X_2,X_3]$ and the only additional possibly nonzero Lie products of vectors $X_1,\ldots X_6$ are

$$[X_1,X_4] = X_6, [X_2,X_5] = \varepsilon X_6, \quad \varepsilon \in \{-1,0,1\} \tag{3}$$

# Jacobi singularity set for (2,3,5,6) distributions and the case of flat distribution

$S_D = \{\lambda \in W_D : B\,(\pi_* \mathcal{C}(\lambda), \pi_*(\mathcal{C}(\lambda))) = 0\}$, i.e. $C(\lambda)$ is projected to a null line of $B$.
The canonical section of $J^{(4-n)}$ can be taken as

$$\mathcal{E} = \frac{\partial_{u_6}}{B(\pi_* \vec{h}(\lambda), \pi_* \vec{h}(\lambda))}$$

and the generalized Wilczynski invariants $A_i$ with $i = 1, 2$ are in general homogeneous rational functions of degree $2(i+1)$ with denominators being $2(i+1)$st powers of the quadratic polynomial $Q(\lambda) := B\left(\pi_* \vec{h}(\lambda), \pi_* \vec{h}(\lambda)\right)$.

However , if the characteristic distribution $C$ is tangent to the level sets of $Q$ , then $A_1$ and $A_2$ are polynomials. In particular, for the flat distributions with parabolic Tanaka symbol $A_1 = A_2 = 0$ , and with the elliptic or hyperbolic Tanaka symbol

$$A_1 = \frac{74}{175} u_6^4, \quad A_2 = -\frac{178}{15435} \varepsilon u_6^6, \quad \varepsilon = \pm 1.$$

THANK YOU FOR YOUR ATTENTION