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On symmetries of symplectically flat rank 3 distributions

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(based on joint work with
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1H]

Equivalence of distributions (subbundles of tangent bundles) up to the group of diffeomorphisms (of the ambient manifold)

$$\underbrace{D = \{D(q)\}_{q\in M}, D(q)\subset T_qM, \dim D(q)=\ell}_{\text{rank } \ell \text{ distribution}}$$

<u>Goal</u> To construct a canonical frame (for on some bundle over M without fixing Tanaka symbol of the distribution

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Tanaka symbol

$$D^j(q) = D^{j-1}(q) + [D, D^{j-1}](q)$$

$D(q)\subset D^2(q)\subset\ldots\subset D^j(q)\subset\ldots\subset T_qM$ - filtration of T_qM

$$\mathfrak{m}(q) = \underbrace{D(q)}_{\mathfrak{g}^{-1}(q)} \oplus \underbrace{D^2(q)/D(q)}_{\mathfrak{g}^{-2}(q)} \oplus \underbrace{D^3(q)/D^2(q)}_{\mathfrak{g}^{-3}(q)} \oplus \ldots$$

$\downarrow$

graded nilpotent Lie algebra

Universal prolongation of Tanaka's symbol

$$\mathfrak{m} = \bigoplus_{i=-\mu}^{-1} \mathfrak{g}^i \quad \text{- a graded Lie algebra}$$

<u>Def</u> Universal prolongation of $\mathfrak{m}$ is a graded Lie algebra $\mathcal{U}(\mathfrak{m}) = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}^i(\mathfrak{m})$ satisfying the following conditions:

(1) The graded subalgebra $\bigoplus_{i<0} \mathfrak{g}^i(\mathfrak{m})$ of $\mathcal{U}(\mathfrak{m})$ coincides with $\mathfrak{m}$;

(2) for any $x \in \mathfrak{g}^i(\mathfrak{m})$, $i \geq 0$ s.t. $x \neq 0$ there exists $y \in \mathfrak{m}$ s.t. $[x, y] \neq 0$ (i.e. $\operatorname{ad} x|_{\mathfrak{m}} \neq 0$);

(3) $\mathcal{U}(\mathfrak{m})$ is the maximal graded algebra satisfying conditions (1) and (2) above.

$\mathcal{U}(\mathfrak{m})$ is the maximal nondegenerate graded Lie algebra containing $\mathfrak{m}$ as its negative part

Tanaka theorem on prolongation

Assume that D is a distribution with constant symbol m, i.e. symbols $m(x)$ are isomorphic (as graded Lie algebras) to m for any point x.

Assume that $\dim \mathcal{U}(m) < \infty$

$\Updownarrow$

$\exists\, l \geq 0$ s.t. the lth algebraic prolongation g^l of m vanishes

Theorem (Tanaka, 1970) One can assign to D in a canonical way a bundle over M of dimension equal to $\dim \mathcal{U}(m)$, equipped with a canonical frame. Dimension of algebra of infinitesimal symmetries of D is not greater than $\dim \mathcal{U}(m)$. This upper bound is sharp and is achieved iff for distribution locally equivalent to the flat distribution D_m

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Restrictions and disadvantages of Tanaka approach.

All constructions strongly depend on the notion of symbol.

In order to apply this machinery to all bracket-generating (ℓ, n)-distributions with fixed ℓ and n, one has

- to classify all n-dimensional graded nilpotent Lie algebras with ℓ generators – hopeless task
- to generalize the Tanaka prolongation procedure to distributions with nonconstant symbol, because the set of all possible symbols may contain moduli.

Alternative approach - 2nd symplectification procedure

It consists of the reduction of the equivalence problem for distributions to the extrinsic differential geometry of curves of flags of isotropic and coisotropic subspaces in a linear symplectic space

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It gives an explicit unified construction of canonical frames for huge classes of distributions, avoiding classification of Tanaka symbols and the possible presence of moduli in the set of Tanaka symbols

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The origin - Optimal Control Theory

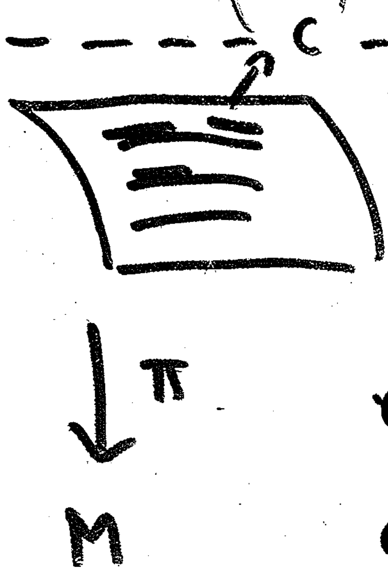
Sketch of symplectification procedure

Step 1 Distingish a special submanifold W_D of $\mathbb{P}T^*M$ endowed with the characteristic 1-foliation (the foliation of abnormal extremals)

For example, for rank $D=2$: $W_D = (D^2)^\perp \setminus (D^3)^\perp$

for rank $D=3$: $W_D = D^\perp \setminus (D^2)^\perp$

Here $(D^j)^\perp = \{(p,q) \in \mathbb{P}T^*M : p(v)=0 \ \forall v \in D^j(q)\}$



W_D - endowed with even contact structure $\tilde{\Delta}$;

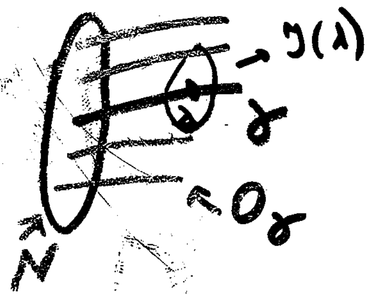
C - characteristic line distribution on $\tilde{\Delta}$ (the Cauchy characteristic distrib of $\tilde{\Delta}$, $[C, \tilde{\Delta}] \subset \tilde{\Delta}$

$$\mathcal{J}(\lambda) = \{v \in T_\lambda W_D : \pi_* v \in D(\pi(\lambda))\}$$

Jacobi curves of abnormal extremals

Let γ be a segment of an abnormal extremal.

O_γ be a neighbourhood of γ in $\mathbb{P}W_D$ such that the factor



$$N = O_\gamma \Big/ \begin{pmatrix} \text{the characteristic} \\ \text{one-foliation} \end{pmatrix}$$

is a well defined smooth manifold.

Let $\varphi : O_\gamma \to N$ be the canonical projection to the quotient

$\Delta := \varphi_* \tilde{\Delta}$ is a contact distribution on N

$$\forall \lambda \in \gamma \qquad F_\gamma(\lambda) := \underbrace{\varphi_* (J(\lambda))}_{\text{coisotropic subspace}} \subset \Delta(\gamma)$$

The curve $\lambda \to F_\gamma(\lambda)$, $\lambda \in \gamma$ is a curve of coisotropic subspaces of $\Delta(\gamma) \subset T_\gamma N$ the Jacobi curve of the abnormal extremal γ.

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- Any invariant of the Jacobi curve F_γ w.r.t. the action of (Conformal) Symplectic Group on the set of coisotropic subspaces of $\Delta(\gamma)$ produces an invariant of the distribution D

reduction to the geometry of curves of coisotropic subspaces of a linear symplectic group

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- The canonical bundles of moving frames associated with Jacobi curves

$\Downarrow$

the canonical frame for D itself on certain fiber bundle over $\mathbb{P}W_D$

The curve of symplectic flags associated with $\lambda \to F_\gamma(\lambda)$

Let $\Gamma(F_\gamma)$ be the space of all smooth sections of $\bigcup_{\lambda\in\gamma} F_\gamma(\lambda)$ (considered as a vector bundle over γ)

1) For $i \geq 0$ set

$$F_\gamma^{-i}(\lambda) := \operatorname{span}\left\{ \frac{d^j}{dt^j}\, \ell(\varphi(t))\Big|_{t=0} : \ell\in\Gamma(F_\gamma),\ 0\leq j<i \right\}$$

where $\varphi: \mathbb{R}\to\gamma$ is a parametrization of γ, $\varphi(0)=\lambda$. In particular, $F_\gamma^{-1}(\lambda) = F_\gamma(\lambda)$

2)

$$F_\gamma^{i}(\lambda) := \begin{cases} \left(F_\gamma^{-i-1}(\lambda)\right)^{\angle} & \text{if } F_\gamma(\lambda) \text{ is a proper coisotropic} \\ \left(F_\gamma^{-i-2}(\lambda)\right)^{\angle} & \text{if } F_\gamma(\lambda) \text{ is Lagrangian} \end{cases}$$

the case of rank 2 distrib.

Symbols of curves of flags

$$\dots \subset F_\gamma^{(\nu)} \subseteq \dots \subseteq \underbrace{F_\gamma^{(0)}}_{\text{isotropic}} \subset \underbrace{F_\gamma^{(-1)} \subseteq F_\gamma^{(-2)} \subseteq \dots \subseteq F_\gamma^{(-\nu)} \subseteq \dots}_{\text{coisotropic}}$$

$$Gr^i(\lambda) := F_\gamma^{(i)}(\lambda) \big/ F_\gamma^{(i+1)}(\lambda)$$

The corresponding graded space $\bigoplus Gr^i(\lambda)$ is endowed with the natural conformal symplectic structure induced from the conformal symplectic structure on Δ

The tangent vector to the refined Jacobi curve at a point corresponding to λ can be identified with a line

$$s_\lambda \subset csp\Big(\bigoplus_{i\in\mathbb{Z}} Gr^i(\lambda)\Big) \text{ of degree } -$$

i.e. s.t. $s_\lambda(Gr^i(\lambda)) \subset Gr^{i-1}(\lambda)$

s_λ - the symbol of the Jacobi curve at λ.

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Geometry of curves of flags of isotropic/coisotropic subspaces with constant symbol $s \in csp(\oplus X^i)$

Similar theory to Tanaka

- Universal prolongation of the symbol s is the largest graded subalgebra of $csp(\oplus X^i)$ containing s as its negative part.

$$U_F(s) = \bigoplus_{i \geq -1} u^i(s), \quad u^{-1}(s) = s$$

Explicit construction recursively:

$$u^i(s) = \{ A \in scp_i(\bigoplus_j X^j) : [A, \delta] \in u^{i-1}(s) \}_{\delta \in s}$$

- Flat curve with symbol s (of type s)

Take the corresponding filtration $\{V^i\}_{i \in \mathbb{Z}}$, $V^i = \bigoplus_{j \geq i} X^j$

The flat curve of type s is $t \to \{e^{t\delta} V^i\}_{i \in \mathbb{Z}}$ $_{\delta \in s}$

$\mathcal{U}_F(s) \sim$ the algebra of infinitesimal symmetries (in $csp(\oplus X^i)$) of the flat curve of type s

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Thm (Doubrov - Zelenko)
To a curve of flags of isotropic/coisotropic subspaces with constant symbol s one can assign in a canonical way a bundle of moving frames of dimension equal to $\dim \mathcal{U}_F(s)$

Rem This results can be generalized to natural classes of curves (and submanifolds) in arbitrary parabolic homogeneous spaces G/P (and more general homogeneous spaces if the Lie algebra of G has fixed grading)

Jacobi symbols of distributions

Finiteness of the set of symbols, up to isomorphism + classification of symplectic symbols

$\Downarrow$

For a generic point $q \in M$ there exists a neighb. U s.t. the symbols of Jacobi curves of abnormal extremals through a generic point of $\mathbb{P}W_D$ over U are isomorphic to one symbol

$$s \in csp_{-1}(\underbrace{\oplus X^i}_{\text{fixed graded symplectic space } V := \oplus X})$$

$\downarrow$

Jacobi symbol of the distribution D at q

New formulation:

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Instead of constructing canonical frames for distributions according to their Tanaka symbols to do it according to their Jacobi symbols, which is much coarser characteristic.

Distributions of maximal class:

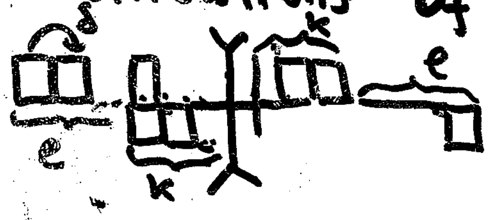
Jacobi curve of a generic abnormal extremal γ does not belong to proper subspaces of $\Delta(\gamma)$

(2, n) - distributions of maximal class - unique Jacobi symbol

$\underbrace{\qquad\qquad}_{2(n-3)\text{ Boxes}}$

δ is the right shift operator → indecomposable symbols

(3, n) - distributions of maximal class



$n = 2k + e + 4$

δ is the right shift operator

From canonical moving frames for Jacobi curves to canonical frames for distributions

Bield the following graded Lie algebra

$$B(s) = \underbrace{\overbrace{\underset{\text{1-dim}}{\eta}}^{\mathfrak{g}^{-2}} \oplus \overbrace{\left(\oplus X^i\right)}^{\mathfrak{g}^{-1}}}_{\substack{\text{Heisenberg algebra - the} \\ \text{Tanaka symbol of the contact} \\ \text{distribution}}} \oplus \overbrace{u_F(s)}^{\mathfrak{g}^{0}}$$

Let $U_T(B(s))$ be the Tanaka universal prolongation of $B(s)$ (i.e. the maximal nondegenerate graded Lie algebra, containing $B(s)$ as its nonpositive part)

Theorem
(Doubrov, Zelenko)

If D is a distribution with Jacobi symbol s, rank $D = 2$ or rank D is odd, and $\dim \mathcal{U}_T(B(s)) < \infty$, then there exists a canonical frame for D on a manifold of dimension equal to $\dim \mathcal{U}_T(B(s))$. In particular, the algebra of infinitesimal symmetries of a distribution D with Jacobi symbol δ is $\leq \dim \mathcal{U}_T(B(s))$. Moreover, there exists a distribution with Jacobi symbol s such that its algebra of infinitesimal symmetries is isomorphic to $\mathcal{U}_T(B(s))$. – symplectically flat distribution with Jacobi symbol

Symplectically flat distributions with the Jacobi symbol s

$$U_T(B(s)) = \underbrace{\eta \oplus V}_{\text{Heisenberg}} \oplus U_F(s) \oplus \bigoplus_{i>0} g^i$$

$$V = \bigoplus_{j\in\mathbb{Z}} X_j$$

$$p(s) = \Big(\bigoplus_{j\geq 0} X_j\Big) \oplus \underbrace{U_F^+(s)}_{\substack{\text{nonnegative}\\ \text{part of } U_F^+(s)}} \oplus \Big(\bigoplus_{i>0} g^i\Big)$$

Symplectic flat distributions with the Jacobi symbol s is the distribution on the homogeneous space corresponding to the pair $(U_T(B(s)), p(s))$ and the subspace

$$\Big(\bigoplus_{j\geq -1} X_j\Big) \oplus U_F(s) \oplus \Big(\bigoplus_{i>0} g^i\Big)$$

in $U_T(B(s))$

How to describe effectively

$$U_T(\underbrace{\eta \oplus V}_{\text{Heisenberg}} \oplus u_F(\underset{\sim}{s})) \ ?$$

Jacobi symbol

1) Description of $u_F(s)$

$$u_F(s) = \underbrace{u_{-1}}_{s} \oplus u_0 \oplus \ldots \subset g = csp(V)$$

Morozov thm: $\forall x \in s \ \exists \ h \in u_0, \ y \in u_1$

s.t. $\langle x, h, y \rangle \sim sl_2$

Then $u_F(s) = sl_2 \ltimes \overset{sl_2\text{-module}}{\widetilde{n}}$, where n is the largest ideal of $u_F(s)$ concentrated in the non-negative part

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n is also called the non-effective ideal of the algebra of infinites. symm. of the flat curve, corresponding to the subgroup of thy symmetry group, acting trivially on the fla[t curve]

The case of rank 2 distributions of maximal class on n-dim manifold

- (Refined) Jacobi curves are curves of complete flags consisting of all osculating subspaces of $\underbrace{\text{the curve of its 1-dim subspaces}}_{\text{a curve in projective space}}$

- Only one Jacobi symbol S^2_n - one row diagram

$\underbrace{\square\square\ \cdots\ \cdots\ \square}_{2(n-3)\text{ boxes}}$ δ is a right shift

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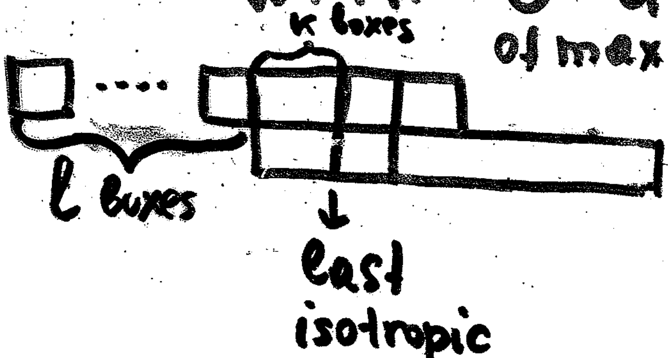
- The flat curve with symbol S^2_n is a curve of complete flags consisting of all osculating subspaces of the rational ~~normal~~ curve in $\mathbb{P}^{2n-7}$ $(t \to [1 : t : \ldots : t^{2n-7}])$

- $U_F(S)$ = the image of the irreducible embedding of $\underline{gl_2}$ into $gl_{\underline{2n-6}}$ $(\underline{so}\ \underline{n} = \mathbb{R})_{\text{Doubrov-Z } 2006}$

$U_T(B(S^2_5)) = G_2$ - $\underset{1910}{\text{Cartan}}$ $U_T(B(S^2_n)) =_{n\geq 6} B(S^2_n)$

The case of rank 3 distributions with 6-dim. square ($\dim D^2 = 6$) of maximal class



V_E - the first row

V_F - the second row

$V = V_E \oplus V_F$

The set of Jacobi symbols of rank 3 distributions is encoded by a pair (k, ℓ)

$k \geq 1 \iff \dim D^2 = 6$ $\qquad S := S_{k,\ell}$

V_E & V_F are irreducible sl_2-modules

$\dim V_E = \dim V_F = \underbrace{2k+\ell+1}_{r}$, $V_E^* \simeq V_F$

Non-effective ideal $\mathfrak{n} \subset csp(V_E \oplus V_F)$

Consider the case $\ell > 0$ (non-rectangular diagrams)

$\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$, where $\mathfrak{n}_1 \subset V_E \otimes V_E^* + V_F \otimes V_F$

and $\mathfrak{n}_2 \subset Symm^2(V_E)$ ($\subset V_E \otimes V_E$ under the identification $V_F^* \sim V_E$)

$\mathfrak{n}_1 = \mathrm{span}\langle \mathrm{Id}, Z \rangle \subset \mathfrak{u}_0$

$Z|_{V_E} = \mathrm{Id}|_{V_E} \quad Z|_{V_F} = -\mathrm{Id}|_{V_F}$

Description of n_2 via sl_2-representations

$$\underbrace{\mathrm{Symm}^2(V_E)}_{\substack{V_E:\ r\text{-dim. irreducible} \\ sl_2\text{-module}}} = \underbrace{\Pi_{2r-2} \oplus \Pi_{2r-6} \oplus \ldots \oplus \begin{matrix}\Pi_2 \\ \text{or} \\ \Pi_0\end{matrix}}_{\text{decomposition onto irreducible } sl_2\text{- module}}$$

Here Π_i is the irreducible sl_2- modules of dimension $i+1$

Since $n_2 \subset$ non-negative part of $U_F(s\ldots)$

$\Downarrow$

$$n_2 = \underset{\substack{\downarrow \\ \text{the shift in the diagram}}}{\Pi_{2e}} \oplus \Pi_{2e-4} \oplus \ldots \oplus \begin{matrix}\Pi_2 \\ \text{or} \\ \Pi_0\end{matrix}$$

Description of n_2 via geometry of rational normal curves

$r = 2k + \ell +$

Fix a coordinate system $(x_1, \dots, x_r, p_1, \dots, p_r)$ in the symplectic space of V such that the symplectic form ω has the form

$$\omega = dx_1 \wedge dp_r - dx_1 \wedge dp_{r-1} + \dots + (-1)^{r+1} dx_r \wedge dp_1$$

$$V_E = \left\langle \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_r} \right\rangle$$

$$V_F = \left\langle \frac{\partial}{\partial p_1}, \dots, \frac{\partial}{\partial p_r} \right\rangle$$

$sp(V_e \oplus V_f) \underset{\text{Hamiltonian formalisms}}{\sim}$ all quadratic polynomials in $\mathbb{R}[x_i, p_j]$

Then

$n_2 \subset Symm^2(V_E) \sim$ degree 2 polynomials in $x_1, \dots, x_r$

Let $[x_1 : \dots : x_r]$ homogeneous coordinates in $\mathbb{P} V_E$; C be the normal rational curve, the image of Veronese embedding

$$\mathbb{P}^1 \to \mathbb{P}^{r-1}, \quad [s:t] \to [s^{r-1} : s^{r-2}t : \dots : t^{r-1}]$$

Description of $\mathfrak{n}_2$ via geometry of rational normal curves (continued)

C - the rational normal curve in $\mathbb{P}V_E$;

$\mathcal{T}^m C$ - the mth tangential developable variety of C (the union of all mth osculating subspaces to C). $(\mathcal{T}^0 C = C)$

Then for $\cancel{s} = S_{k,\ell}$

$$\mathfrak{n}_2 = \Pi_{2\ell} \oplus \Pi_{2\ell-4} \oplus \dots \oplus \begin{matrix}\Pi_2 \\ \text{or} \\ \Pi_0\end{matrix} \sim$$

(Fulton-Harris)

$\sim$ the space of degree 2 polynomials in $x_1, \dots, x_r$ vanishing on $\mathcal{T}^{k-1} C$

$$U_F(s) = \mathfrak{sl}_2 \, \lambda(\underset{\frown}{\mathfrak{n}_1} \oplus \mathfrak{n}_2)$$

2-dim subspace of u_0

How to describe

$$U_T\Big(\underbrace{\overbrace{\eta \oplus V}^{\text{Heisenberg}} \oplus U_F(s_{\kappa,e})}_{B(s_{\kappa,e})}\Big)$$

that is the symmetry algebra of the symplectically flat with Jacobi symbol $s_{\kappa,e}$

Secant variety: Given an algebraic variety $X \subset \mathbb{P}V_E$ let $S_m(X)$ be the mth secant variety of X which is the algebraic closure of the union of $(m-1)$-planes in $\mathbb{P}V_E$ passing through m points of X, $S_1(X) = X$.

The mth Tanaka prolongation of $B(s_{\kappa,e})$ ~ the space of all degree m+ polynomials that vanish on $S_{m+1}(J^{\kappa-1}(C))$

The mth Tanaka prolongation of $B(s_{k,e})$ ~ the space of all degree $m+2$ polynomials that vanish on $\underbrace{S_{m+1}(J^{k-1}(C))}$

(m+1)st secant variety of (k-1)th tangential developable variety of the rational normal curve

at least the rth secant of C is the ~~whole~~ $\mathbb{P}V_E$

$\Downarrow$

$U_T(\eta \oplus V \oplus U_F(s_{k,e}))$ is of finite type for $k \geq 1$ ($\Leftarrow$ $\dim D^2 = 6$, D is of maximal class)

In the case $k=1$

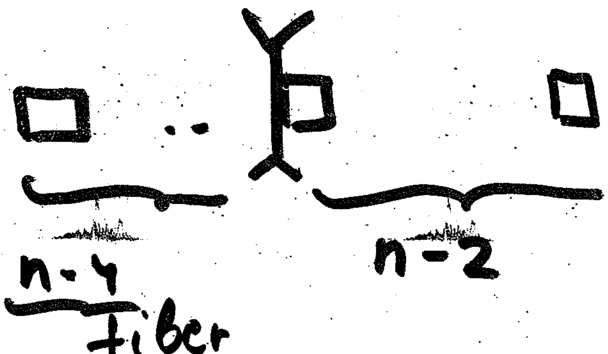
$$(*) \quad \begin{pmatrix} x_1 & x_2 & x_3 & \dots & x_{r-d} \\ x_2 & x_3 & x_4 & \dots & x_{r-d+1} \\ \vdots & & & & \\ x_{d+1} & x_{d+2} & x_{d+3} & \dots & x_r \end{pmatrix} \quad n+1 \leq d \leq r-n-2$$

$\boxed{\text{Harris Algebraic geom}}$

The space of degree $n+1$ polynomials vanishing on $S_n(\ldots$ is spanned by rank $n+1$ minors of matrix $(*)$

More explicit model for symplectically flat distributions with Jacobi symbol s can be deduced from the diagrams corresponding to s

- rank 2 distributions on n-dim



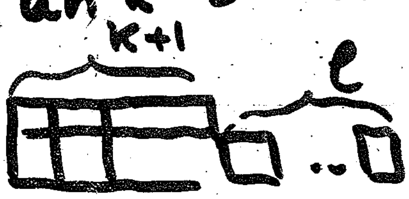
$\langle H, X_1, \dots, X_{n-2}, N\rangle$ - Lie alg with non-zero products

$[H, X_i] = X_{i+1}$

$[X_1, X_2] = N$

$D = \langle H, X_1\rangle$

- rank 3- distrib with Jacobi symbol $s_{k\ell}$



$$\left\langle \begin{matrix} X_1, \dots, X_{k+1} \\ Y_1, \dots, Y_{k+1}, \dots Y_{k+\ell+1} \\ H, N \end{matrix} \right\rangle$$

$[H, X_i] = X_{i+1}$
$[H, Y_i] = Y_{i+1}$

$[X_1, Y_1] = N$, $D = \langle X_1, Y_1, H\rangle$