Homework assignment 6 Spring 2016 - MATH622

**due May 3 2016 at the beginning of class (I will give an extra class at this day)**

**Solve any 5 out of 6 first problems below, you also can get up to 140 points for solving all these 6 problems and the bonus problem 7**

1. A volume form $\Omega$ on an $n$-dimensional manifold $M$ is an everywhere nonvanishing smooth differential $n$ form on $M$. Given a volume form $\Omega$ on $M$, one can define the divergence of a vector field X (w.r.t. the volume form) as the unique scalar-valued function, denoted by $\text{div}X$, satisfying
$$L_X\Omega = (\text{div}X)\Omega.$$
Prove that if $M = \mathbb{R}^3$ with coordinates $(x, y, z)$ and $\Omega = dx \wedge dy \wedge dz$, then this coincides with the definition of the divergence of a vector field given in the Multivariable Calculus.

2. (a) Solve Problem 11 page 78;
   (b) Let $\omega$ be a smooth 1-form on a smooth manifold $M$. A smooth positive function $\mu$ on some subset $U \subset M$ is called an integrating factor if $\mu\omega$ is exact on $U$, i.e. there exists a smooth function $f$ on $U$ such that $d f = \mu\,\omega$. If $\omega$ is nowhere-vanishing, show that $\omega$ admits an integrating factor in a neighborhood of each point if and only if $d\omega \wedge \omega = 0$.

3. (a) Show that the Example 3.3 (i) on page 83 of Warner is a Lie group.
   (b) Show that the Example 3.5 (e) on page 84 of Warner is a Lie algebra and it is isomorphic to the Lie algebra of $3 \times 3$ skew-symmetric matrices.
   (c) Problem 17, page 135 of Warner.

4. Let $G$ be Lie group
   (a) Let $F : G \times G \to G$ denote the multiplication map. Identify the space $T_{(e,e)}(G \times G)$ with $T_eG \oplus T_eG$ by
$$v \in T_{(e,e)}(G \times G) \mapsto \Big(d\pi_1(v), d\pi_2(v)\Big)$$
   where $\pi_1$ and $\pi_2$ are projections of $G\times G$ to the first and the second components, respectively. Show that $dF_e : T_eG \oplus T_eG \to T_eG$ is given by $dF(X, Y) = X + Y$.
   (b) Let $J : G \to G$ denote the inversion map. Show that $dJ_e : T_eG \to T_eG$. is given $dJ_e(X) = -X$.

5. Problem 15 page 159 of Warner.

6. Problem 16 (a)-(c) page 159 of Warner.

7. (**bonus 20 points**) Prove that if $G$ is a group and a differentiable manifold such that the multiplication map $G \times G \mapsto G$ define by $(\sigma, \tau) \mapsto \sigma\tau$ is smooth, then the map $\tau \mapsto \tau^{-1}$ is smooth (this shows that we can equivalently define a Lie group as a group and a differentiable manifold with a smooth multiplication).