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On geometry and symmetries of nonholonomic distributions and curves of flags.

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Two problems in local differential geometry

I) Equivalence of vector distributions (subbundles of tangent bundles) with respect to the group of diffeomorphisms of M

$$\underbrace{D = \{D(q)\}_{q\in M},\ D_q \subset T_q M,\ \dim D(q) = \ell}_{\text{rank } \ell \text{ distribution on } M}$$

The natural filtration of TM – the weak derived flag: $D^1(q) := D(q)$

$D^j(q) = D^{j-1}(q) + [D, D^{j-1}](q) =$

$= \operatorname{Span}\left\{\begin{array}{l}\text{all Lie brackets of length} \leq j\\ \text{of section of } D \text{ evaluated}\\ \text{at the point } q\end{array}\right\}$

$D(q) \subset D^2(q) \subset \ldots \subset D^j(q) \subset \ldots$ in $T_q M$

Bracket-generating distribution: $\forall q \in M$ $\exists\ \mu(q)$ s.t. $D^{\mu(q)}(q) = T_q M$

II) Equivalence of curves of flags of a linear space W with respect to a Lie subgroup G of $GL(W)$

Given integers $0 = k_0 \le k_1 \le \ldots \le k_\mu = \dim W$ let $F_{k_1,\ldots,k_{\mu-1}}(W)$ be the manifold of all flags $0 = \Lambda_0 \subset \Lambda_{-1} \subset \ldots \subset \Lambda_{-\mu} = W$, where $\dim \Lambda_{-i} = k_i$, $0 \le i \le \mu$

$GL(W)$ acts naturally on $F_{k_1,\ldots,k_\mu}(W)$

Let $\mathcal{O}$ be an orbit in $F_{k_1,\ldots,k_\mu}(W)$ w.r.t. the action of G

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We consider (unparametrized) curves in $\mathcal{O}$ <u>compatible with respect to differentiation</u>:

(*) $t \to \{0 = \Lambda_0(t) \subset \Lambda_{-1}(t) \subset \ldots \subset \Lambda_{-\mu}(t) = W\}$ - parametrized somehow

In general, $\frac{d}{dt}\Lambda_i(t) \subset \mathrm{Hom}\left(\Lambda_i(t), W/\Lambda_i(t)\right)$

The curve (*) is called <u>compatible w.r.t. differentiation</u> if $\boxed{\frac{d}{dt}\Lambda_i(t) \in \mathrm{Hom}\left(\Lambda_i(t), \Lambda_{i-1}(t)/\Lambda_i(t)\right)}$

Tanaka theory of filtered structures (Brief review)

Set $D^{-i}(q) := D^i(q)$

$$D(q) = D^{-1}(q) \subset D^{-2}(q) \subset \ldots \subset D^{-\mu+1}(q) \subset D^{-\mu}(q) = T_q M$$

Let $\begin{cases} g^i(q) = D^i(q)/D^{i+1}(q), & i < -1 \\ g^{-1}(q) = D^{-1}(q) \end{cases}$

The corresponding graded space

$$m(q) = g^{-1}(q) \oplus g^{-2}(q) \oplus \ldots \oplus g^{-\mu}(q)$$

is endowed naturally with the structure of a graded nilpotent Lie algebra:

Let $\pi_i : D^i(q) \to D^i(q)/D^{i+1}(q)$ be the canonical projection; $Y_1 \in g^i(q)$, $Y_2 \in g^j(q)$

$\widetilde{Y}_1$ is a section of D^i s.t. $\pi_i(\widetilde{Y}_1(q)) = Y_1$

$\widetilde{Y}_2$ is a section of D^j s.t. $\pi_j(\widetilde{Y}_2(q)) = Y_2$

$$\boxed{[Y_1, Y_2] \overset{def}{=} \pi_{i+j}([\widetilde{Y}_1, \widetilde{Y}_2](q))}$$

$m(q)$ is called the symbol of D at q

Universal prolongation of Tanaka's symbol

$m = \bigoplus_{i=-\mu}^{-1} g^i$ - a graded Lie algebra

Def Universal prolongation of m is a graded Lie algebra $U(m) = \bigoplus_{i \in \mathbb{Z}} g^i(m)$ satisfying the following conditions:

(1) The graded subalgebra $\bigoplus_{i<0} g^i(m)$ of $U(m)$ coincides with m;

(2) for any $x \in g^i(m)$, $i \geq 0$ s.t. $x \neq 0$ there exists $y \in m$ s.t. $[x, y] \neq 0$ (i.e. $\operatorname{ad} x|_m \neq 0$);

(3) $U(m)$ is the maximal graded algebra satisfying conditions (1) and (2) above.

$U(m)$ is the maximal nondegenerate graded Lie algebra containing m as its negative part

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Realization of universal prolongation

$$m = \bigoplus_{i=-\mu}^{-1} g^i$$

$$g^0(m) = \left\{ f \in \mathrm{End}(m) : \begin{array}{l} f([v_1, v_2]) = [f(v_1), v_2] + \\ + [v_1, f(v_2)],\ f(g^i) \subseteq g^i \\ \qquad\qquad \forall\, i < 0 \end{array} \right\}$$

$\downarrow$

the algebra of all derivations of m preserving grading

$m \oplus g^0$ is a graded Lie algebra:

$[f, v] := f(v), \quad f \in g^0, \quad v \in m$

The first algebraic prolongation of m:

$$g^1(m) = \left\{ f \in \bigoplus_{i<0} \mathrm{Hom}(g^i, g^{i+1}) : \begin{array}{l} f([v_1, v_2]) = [f(v_1), v_2] + [v_1, f(v_2)] \\ \qquad \forall\, v_1, v_2 \in m \end{array} \right\}$$

Higher order algebraic prolongations of m

by induction

Induction: Assume that $\mathfrak{g}^i$ are already constructed for $0 \le i < k$. Then

$$\mathfrak{g}^k(\mathfrak{m}) := \left\{ f \in \bigoplus_{i<0} \operatorname{Hom}(\mathfrak{g}^i, \mathfrak{g}^{i+k}) : \begin{array}{l} f([v_1,v_2]) = [f(v_1),v_2] + \\ + [v_1, f(v_2)] \\ \forall\, v_1, v_2 \in \mathfrak{m} \end{array} \right\}$$

$\downarrow$

kth algebraic prolongation of $\mathfrak{m}$

The flat distribution of constant symbol $\mathfrak{m}$

Let $M(\mathfrak{m})$ be the simply connected Lie group with the Lie algebra $\mathfrak{m}$;
e be the identity of $M(\mathfrak{m})$.
The flat (or standard) distribution $D_{\mathfrak{m}}$ of type $\mathfrak{m}$ is a left-invariant distribution on $M(\mathfrak{m})$ such that $D_{\mathfrak{m}}(e) = \mathfrak{g}^{-1}$

If $\dim \mathcal{U}(\mathfrak{m}) < \infty$, then $\mathcal{U}(\mathfrak{m}) \sim$ the algebra of infinitesimal symmetries of the flat distribution $D_{\mathfrak{m}}$ of type $\mathfrak{m}$

Tanaka theorem on prolongation

Assume that D is a distribution with constant symbol m, i.e. symbols $m(x)$ are isomorphic (as graded Lie algebras) to m for any point x.

Assume that $\dim \mathcal{U}(m) < \infty$

$\Updownarrow$

$\exists\, \ell \geq 0$ s.t. the ℓth algebraic prolongation g^{ℓ} of m vanishes

Theorem (Tanaka, 1970) One can assign to D in a canonical way a bundle over M of dimension equal to $\dim \mathcal{U}(m)$, equipped with a canonical frame. Dimension of algebra of infinitesimal symmetries of D is not greater than $\dim \mathcal{U}(m)$. This upper bound is sharp and is achieved iff for distribution locally equivalent to the flat distribution D_m

4HJ 8nd

Restrictions and disadvantages of Tanaka approach.

All constructions strongly depend on the notion of symbol.

In order to apply this machinery to all bracket-generating (ℓ, n)-distributions with fixed ℓ and n, one has

- to classify all n-dimensional graded nilpotent Lie algebras with ℓ generators - hopeless task
- to generalize the Tanaka prolongation procedure to distributions with nonconstant symbol, because the set of all possible symbols may contain moduli

5H3 2nd

Alternative approach - symplectification procedure

It consists of the reduction of the equivalence problem for distributions to the extrinsic differential geometry of curves of flags of isotropic and coisotropic subspaces in a linear symplectic space

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It gives an explicit unified construction of canonical frames for huge classes of distributions, avoiding classification of Tanaka symbols and the possible presence of moduli in the set of Tanaka symbols

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The origin - Optimal Control Theory

10K

Analog of Tanaka theory for extrinsic geometry of curves of flags

W - a linear space, $G \subset GL(W)$

$\mathcal{O} \subset F_{k_1,\dots,k_{\mu-1}}(W)$ - an orbit w.r.t. the action of G

Compatibility of the pair $(G, \mathcal{O})$ w.r.t. grading

Let $\mathfrak{g} \subset \mathfrak{gl}(W)$ be the Lie algebra of G

let $f_0 \in \mathcal{O}$. Then f_0 (as a filtration of W) induces the filtration on $\mathfrak{gl}(W)$ and therefore on $\mathfrak{g}$. Let $gr_{f_0}\mathfrak{gl}(W)$ and $gr_{f_0}\mathfrak{g}$ be the corresponding graded spaces

Under a natural identification $gr_{f_0}\mathfrak{gl}(W) \sim \mathfrak{gl}(gr_{f_0}W)$

$gr_{f_0}\mathfrak{g}$ is a subalgebra of $\mathfrak{gl}(gr_{f_0}W)$

(here $gr_{f_0}W$ is the graded space corresponding to the filtration f_0 of W)

In general $gr_{f_0}\mathfrak{g}$ is no isomorphic to $\mathfrak{g}$

The pair $(G, \mathcal{O})$ is called compatible w.r.t. the grading if for some (and therefore any) $f_0 \in \mathcal{O}$,

$$f_0 = \{ 0 = \Lambda_0 \subset \Lambda_{-1} \subset \dots \subset \Lambda_{-\mu} = W \}$$

there exists a map $\mathcal{J} : gr_{f_0} W \to W$ such that:

1) $\mathcal{J}(\Lambda^i / \Lambda_{i+1}) \subset \Lambda_i$, $-\mu \leq i \leq -1$;

2) $\mathcal{J}$ conjugates the Lie algebras $gr_{f_0} g$ and g, i.e.

$$g = \{ \mathcal{J} \circ x \circ \mathcal{J}^{-1} : x \in gr_{f_0} g \}$$

defines the grading on g (up to a conjugation)

So we can start with given grading of g and try to construct a flag f_0 in W s.t. $gr_{f_0} g$ is conj to g. The orbit $\mathcal{O}_{f_0}$ - compatible with grading of g.

For example, if W is equipped with a symplectic form ε and $G = Sp(W)$, then an orbit $\mathcal{O}$ is compatible w.r.t. the grading on $\mathfrak{g} = sp(W)$ iff some (and therefore any) flag $f_0 \in \mathcal{O}$ satisfies the following two conditions:

(1) any subspace in the flag f_0 is either isotropic or coisotropic w.r.t. ε;

(2) a subspace belongs to the flag f_0 together with its skew-symmetric complement w.r.t. ε.

For $G = \underbrace{CSp(W)}_{\text{conformal symplectic}}$ - the same conclusion

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A flag f_0 satisfying conditions (1) and (2) is called a symplectic flag.

How to construct orbits compatible with given grading of g? $g=\bigoplus_{i\in\mathbb{Z}} g_i$

Assume that there exist a grading element e (namely $ad_e x = ix, \forall x \in g_i$) and e, as an endomorphism of W, is diagonizable.

Let $\operatorname{Spec} e = \bigsqcup_{j<0} A_j$ s.t.

if $\lambda \in A_j$ and $\lambda + i \in \operatorname{Spec} e$ for some $i \in \mathbb{Z}$ then $\lambda + i \in A_{j+i}$

Let $W = \bigoplus W_j$, where W_j is the invariant subspace, corresponding to the subset A_j of $\operatorname{Spec} e$

Let $f_0 = \{W^j\}_{j\in\mathbb{Z}}$, where $W^j = \bigoplus_{i\geq j} W_i$

Then O_{f_0} is compatible with the grading of g

14K

Let $(\mathfrak{G}, \mathcal{O})$ be compatible w.r.t. the grading

Fix $f_0 \in \mathcal{O}$, $f_0 = \{0 = \Lambda_0 \subset \Lambda_{-1} \subset \ldots \subset \Lambda_{-\mu} = W\}$

$V = gr_{f_0} W$, $V_i = \Lambda_i / \Lambda_{i-1}$, $V = \oplus V_i$

Fix $J : V \to W$ s.t. $J(V_i) \subset \Lambda_i$

and J conjugates $gr_{f_0} g$ and g

$\underline{G} := \{ J^{-1} \circ A \circ J : A \in \mathfrak{G} \}$

$\underline{g} := gr_{f_0} g = \oplus \underbrace{\underline{g}_i}_{\text{degree } i \text{ endomorphisms of } \underline{g}}$

$V^j = \bigoplus_{i=j}^{-1} V_i$, $\{V^j\}_{j=-\mu}^{-1}$ is a filtration of V

$\underline{G}_+$ - a subgroup of $\underline{G}$ preserving $\{V^j\}_{j=-\mu}^{-1}$

$\hat{P}$ be the bundle over $\mathcal{O}$ with the fiber $\hat{P}_L$ over a point L consisting of all isomorphisms $A : V \to W$ s.t. 1) $A(V^i) \subset L_i$;

2) $\mathfrak{G} = \{ A \circ X \circ A^{-1} : X \in \underline{G} \}$

3) $A \circ J^{-1} \in \mathfrak{G}$

$\hat{P} \sim \underline{G}$

$\mathcal{O} = \underline{G} / \underline{G}_+$

$\hat{P}$ is a $\underline{G}_+$ - principal bundle over $\mathcal{O}$.

15K

Compatibility w.r.t. differention and the symbol

$t \to \Lambda(t) := \{ 0 = \Lambda_0(t) \subset \Lambda_{-1}(t) \subset \dots \subset \Lambda_{-\mu}(t) = W \}$

$$\boxed{\forall_i \quad \frac{d}{dt}\Lambda_i(t) \in \mathrm{Hom}\left(\Lambda_i(t), \Lambda_{i-1}(t) / \Lambda_i(t)\right)}$$

$\Downarrow$ factors through a map

$$\delta_t \in \bigoplus \mathrm{Hom}\left(\Lambda_i(t) / \Lambda_{i+1}(t), \Lambda_{i-1}(t) / \Lambda_i(t)\right)$$

– a tangent vector to $\Lambda(\cdot)$ at t

a degree -1 endomorphism of $\mathrm{gr}_{\Lambda(t)} W$ is well defined up to a multiplication by a nonzero constant (a reparametrization is allowed)

Then $\delta_t \in \mathrm{gr}_{\Lambda(t)}\, g$ for any t

The group $\underline{G}_+$ acts naturally on $\underline{g}_{-1}$

$$x \to \left((\mathrm{Ad}\, A) x \right)_{-1}$$

The orbit of the line

$$s = \mathbb{R}\left(A_t^{-1} \circ \delta_t \circ A_t\right)_{-1}, \quad A_t \in P_{\Lambda(t)} \quad \subset \underline{g}_{-1}$$

with respect to this action is called the symbol of the curve $\Lambda(\cdot)$ at the point $\Lambda(t)$ with respect to G.

If $\underline{G}$ is semisimple (reductive) then the set of orbits w.r.t. to the action of $\underline{G}_+$ on $\underline{g}_{-1}$ is finite

(E. Vinberg, 1976)

We will consider curves of flags with constant symbol m.

17K

Flat curves with given symbol $\underline{s}$ is a curve that is G-equivalent to the curve $t \to \{ \mathcal{J}_0 e^{t\delta} V^i \}_{i=0,-1,\ldots,-\mu}$, $\delta \in S$

Universal prolongation of symbols of curves

Let s be a line in $\underline{g}_{-1}$

Universal prolongation of s is the largest graded subalgebra of $\underline{g}$ containing s as its negative part

$$U_F(s) = \bigoplus_{i \geq -1} u_i(s), \quad u_{-1}(s) = s$$

Explicit construction recursively

$$u_i(s) = \{ x \in \underline{g}_i : [x, s] \subset u_{i-1} \}$$

Geometric interpretation of $U_F(s)$ - isomorphic to algebra of infinitesimal symmetries of the flat curve of type m

Thm (Doubrov - Zelenko)
To a curve of flags $\Lambda(\cdot)$ (compatible w.r.t. differention) with constant symbol s (w.r.t. G) one can assign in a canonical way a bundle of moving frames (a fiber subbundle of $\hat{P}|_{\Lambda(\cdot)}$ endowed with a canonical Ehresmann connection) of the dimension equal to $\dim U_F(s)$.

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Symplectification procedure

Step 1. To distinguish a special submanifold of T^*M endowed with the characteristic 1-foliation (the foliation of abnormal extremals).

Let $T^*M = \{(p,q) : q \in M,\ p \in T_q^*M\}$ be the cotangent bundle; ω be the canonical symplectic form on it;

$$(D^j)^{\perp} = \{(p,q) : p \cdot v = 0 \quad \forall v \in D^j(q)\}$$

be the annihilator of the jth power of D

$$\widetilde{W}_D = \{\lambda \in D^{\perp} : \ker\left(\omega|_{D^{\perp}}(\lambda)\right) \neq 0\}$$

- If rank of D is odd then
$$\widetilde{W}_D = D^{\perp} \quad ;$$

- If $\operatorname{rank} D = 2$ then
$$\widetilde{W}_D = (D^2)^{\perp}.$$

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$\widetilde{W}_D$ is odd dimensional. Define $W_D \subset \widetilde{W}_D$:

$$W_D = \{\lambda \in \widetilde{W}_D : \ker \varepsilon|_{\widetilde{W}_D}(\lambda) \text{ is one-dimensional}\}$$

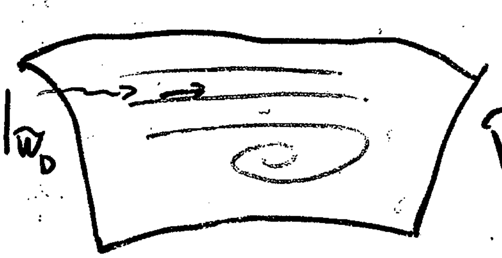
W_D is open and dense in $\widetilde{W}_D$ for generic D

Examples:

- If rank of D is 2, then $W_D = (D^2)^\perp \setminus (D^3)^\perp$
- If rank of D is 3, then $W_D = D^\perp \setminus (D^2)^\perp$

The kernels of $\varepsilon|_{W_D}$ form the <u>characteristic line distribution</u> C_0 on W_D.

The integral curves of C_0 are called <u>(regular) abnormal extremals of D</u>



$$\begin{array}{ccc} \widetilde{W}_D & \hookrightarrow & T^*M \\ \cup & & \downarrow \pi \\ W_D & & M \end{array}$$

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It is more convenient to projectivize the fibers:

$T^*M \to \mathbb{P}T^*M$ $\qquad\qquad$ $W_D \to \mathbb{P}W_D$

Liouville 1-form on T^*M $\to$ contact structure on $\mathbb{P}T^*M$ $\Rightarrow$ even contact structure $\tilde{\Delta}$ on $\mathbb{P}W_D$

$\mathbb{P}\tilde{W}_D$ $\quad$ A pushforward C of C_0 to $\mathbb{P}W_D$ is a well-defined line distribution, which is exactly the Cauchy characteristic distrib. of $\tilde{\Delta}$:

$$[C, \tilde{\Delta}] \subset \tilde{\Delta}$$

Define $\quad \mathcal{J}(\lambda) = \{ v \in T_\lambda \mathbb{P}W_D : \pi_* v \in D(\pi(\lambda)) \}$

$\qquad\qquad V(\lambda) = \{ v \in T_\lambda \mathbb{P}W_D : \pi_* v = 0 \}$ - tangent to the fibers

where $\pi : \mathbb{P}T^*M \to M$ is the canonical proj

$$V + C \subset \mathcal{J}$$

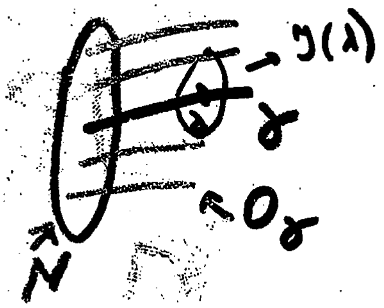
We will work with the distributions $C, V, \mathcal{J}$ instead of the original distribution D.

Step 2: 14 NO 22K

Jacobi curves of abnormal extremals

Let γ be a segment of an abnormal extremal.

O_γ be a neighbourhood of γ in $\mathbb{P}W_D$ such that the factor



$$N = O_\gamma \Big/ \begin{pmatrix} \text{the characteristic} \\ \text{one-foliation} \end{pmatrix}$$

is a well defined smooth manifold.

Let $\varphi : O_\gamma \to N$ be the canonical projection to the quotient

$\Delta := \varphi_* \tilde{\Delta}$ is a contact distribution on N

$$\forall \lambda \in \gamma \qquad F_\gamma(\lambda) := \underbrace{\varphi_*\left(J(\lambda)\right)}_{\text{coisotropic subspace}} < \Delta(\gamma)$$

The curve $\lambda \to F_\gamma(\lambda)$, $\lambda \in \gamma$ is a curve of coisotropic subspaces of $\Delta(\gamma) \subset T_\gamma N$

the Jacobi curve of the abnormal extremal γ.

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The curve of symplectic flags associated with $\lambda \to F_\gamma(\lambda)$

Let $\Gamma(F_\gamma)$ be the space of all smooth sections of $\bigcup_{\lambda\in\gamma} F_\gamma(\lambda)$ (considered as a vector bundle over γ)

1) For $i \geq 0$ set

$$F_\gamma^{-i}(\lambda) := \operatorname{span}\left\{ \frac{d^j}{dt^j}\,\ell(\varphi(t))\Big|_{t=0} : \begin{array}{l}\ell\in\Gamma(F_\gamma)\\ 0\le j<i\end{array}\right\}$$

where $\varphi: \mathbb{R} \to \gamma$ is a parametrization of γ, $\varphi(0)=\lambda$. In particular, $F_\gamma^{-1}(\lambda) = F_\gamma(\lambda)$

2)

$$F_\gamma^{i}(\lambda) := \begin{cases} \left(F_\gamma^{-i-1}(\lambda)\right)^{\angle} & \text{if } F_\gamma(\lambda) \text{ is a proper coisotropic} \\ \left(F_\gamma^{-i-2}(\lambda)\right)^{\angle} & \text{if } F_\gamma(\lambda) \text{ is Lagrangian} \end{cases}$$

the case of rank 2 distrib.

$\lambda \to \{F_\gamma^i(\lambda)\}$ - the refined Jacobi curve of an abnormal extremal γ.

Jacobi symbols of distributions

Finiteness of the set of symbols, up to isomorphism + classification of symplectic symbols

$\Downarrow$

For a generic point $q \in M$ there exists a neighb. U s.t. the symbols of the refined Jacobi curves of abnormal extremals through a generic point of $\mathbb{P}W_D$ over U are isomorphic to one symbol

$$s \subset csp_{-1}(\underbrace{\oplus X^i}_{\text{fixed graded symplectic space } V := \oplus X^i})$$

$\downarrow$

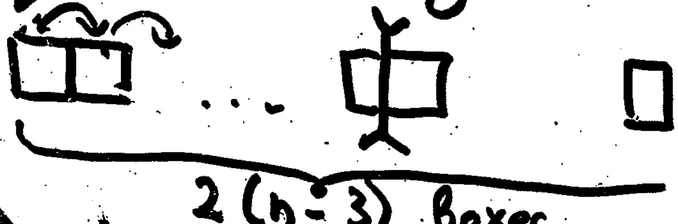
Jacobi symbol of the distribution D at q

New formulation: 25K 14HJ

Instead of constructing canonical frames for distributions according to their Tanaka symbols to do it according to their Jacobi symbols, which is much coarser characteristic.

Distributions of maximal class: Jacobi curve of a generic abnormal extremal γ does not belong to proper subspaces of $\Delta(\gamma)$

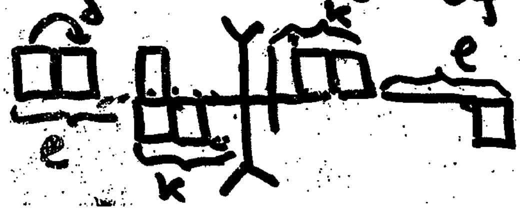
$(2,n)$-distributions of maximal class - unique Jacobi symbol



δ is the right shift operator

indecomposible symbols

$(3,n)$-distributions of maximal class



$n = 2k + \ell + 4$

δ is the right shift operator

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From canonical moving frames for Jacobi curves to canonical frames for distributions

Bield the following graded Lie algebra

$$B(s) = \underbrace{\overbrace{\underset{\downarrow\atop \text{1-dim}}{\eta}}^{g^{-2}} \oplus \overbrace{\underbrace{\left(\bigoplus X^i\right)}_{V}}^{g^{-1}}} \oplus \overbrace{u_F(s)}^{g^{0}}$$

Heisenberg algebra - the Tanaka symbol of the contact distribution

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Let $U_T(B(s))$ be the Tanaka universal prolongation of $B(s)$ (i.e. the maximal nondegenerate graded Lie algebra, containing $B(s)$ as its nonpositive part)

Theorem
(Doubrov, Zelenko)

If D is a distribution with Jacobi symbol s, rank $D = 2$ or rank D is odd, and $\dim \mathcal{U}_T(B(s)) < \infty$, then there exists a canonical frame for D on a manifold of dimension equal to $\dim \mathcal{U}_T(B(s))$. In particular, the algebra of infinitesimal symmetries of a distribution D with Jacobi symbol δ is $\leq \dim \mathcal{U}_T(B(s))$. Moreover, there exists a distribution with Jacobi symbol s such that its algebra of infinitesimal symmetries is isomorphic to $\mathcal{U}_T(B(s))$. – symplectically flat distribution with Jacobi symbol s