On geodesic equivalence of Riemannian metrics and sub-Riemannian metrics on distributions of corank 1

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Abstract

The present paper is devoted to the problem of (local) geodesic equivalence of Riemannian metrics and sub-Riemannian metrics on generic corank 1 distributions. Using Pontryagin Maximum Principle, we treat Riemannian and sub-Riemannian cases in an unified way and obtain some algebraic necessary conditions for the geodesic equivalence of (sub-)Riemannian metrics. In this way first we obtain a new elementary proof of classical Levi-Civita's Theorem about the classification of all Riemannian geodesically equivalent metrics in a neighborhood of so-called regular (stable) point w.r.t. these metrics. Secondly we prove that sub-Riemannian metrics on contact distributions are geodesically equivalent iff they are constantly proportional. Then we describe all geodesically equivalent sub-Riemannian metrics on quasi-contact distributions. Finally we make the classification of all pairs of geodesically equivalent Riemannian metrics on a surface, which proportional in an isolated point. This is the simplest case, which was not covered by Levi-Civita's Theorem.

1 Introduction

Let us recall that two Riemannian metrics on a manifold M are called *geodesically* (or *projective*) *equivalent* at a point $q_0 \in M$, if in some neighborhood of q_0 all their geodesics, considered as unparametrized curves, coincide. The notion of geodesic equivalence can be generalized directly to sub-Riemannian metrics by replacing Riemannian geodesics by normal sub-Riemannian geodesics:

Let D be a bracket-generating (completely nonholonomic) distribution on M. A Lipschitzian curve $\xi(t)$ is called admissible for the distribution D, if it is tangent to D almost everywhere, i.e., $\dot\xi(t) \in D\big(\xi(t)\big)$ a.e.. A sub-Riemannian metric G on D is given by choosing an inner product $G_q(\cdot,\cdot)$ on each subspaces $D(q)$ for any $q \in M$ smoothly w.r.t. q. Let $||\cdot||_q = \sqrt{G_q(\cdot,\cdot)}$ be the corresponding Euclidean norm on $D(q)$. For any admissible curve $\xi : [0,T] \mapsto M$ its length w.r.t. the sub-Riemannian metric G is equal to $\int_0^T ||\dot\xi(t)||_{\xi(t)}\, dt$. Given two points q_1 and q_2 one can look for the curve of minimal length among all admissible curves connecting q_1 with q_2. This problem can be obviously reformulated as a time-minimal control problem (for this one takes into the consideration only admissible curves parametrized by the length). The *sub-Riemannian extremal trajectory* w.r.t. the metric G is the projection to M of a Pontryagin extremal of this problem (which lives in the cotangent bundle T^*M).

In general, Pontryagin extremals can be normal or abnormal: the extremal is called abnormal, if the Lagrange multiplier of the functional is equal to zero, and normal otherwise. The projection of normal (abnormal) Pontryagin extremal is called a *normal (abnormal) sub-Riemannian extremal trajectory*. Any abnormal sub-Riemannian extremal trajectory, considered

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as unparametrized curves, is characterized by distribution D only, but not by the metric on it. Normal sub-Riemannian extremals surely depend on the metric. They can be described in the following simple way: Let $h : T^*M \mapsto R$ satisfies

$$h(p,q) = \frac{1}{2}\big(\max\{p(v) : ||v||_q = 1, v \in D(q)\}\big)^2 \quad q \in M,\, p \in T_q^*M. \tag{1.1}$$

Then the normal sub-Riemannian extremal trajectories are exactly the projections on M of the trajectories of the Hamiltonian system $\dot{\lambda} = \vec{h}(\lambda)$, lying on the $\frac{1}{2}$-level set of h, i.e., on the set $\{\lambda \in T^*M : h(\lambda) = \frac{1}{2}\}$.

Remark 1 *The norm $||\cdot||_q$ on D_q induces the norm on the dual space, which will be denoted also by $||\cdot||_q$. Therefore taking the restriction $p|_{D(q)}$ of some covector $p \in T_q^*M$ one can rewrite (1.1) in the following form*

$$h(p,q) = \frac{1}{2}||p|_{D(q)}||_q^2 \quad q \in M,\, p \in T_q^*M. \tag{1.2}$$

Note that a Riemannian metric is actually the sub-Riemannian metric with $D = TM$ and classical Riemannian geodesics are exactly normal extremal trajectories in this situation (here abnormal extremals do not exist). Note also that, as in Riemannian case, sufficiently small pieces of normal sub-Riemannian extremal trajectories are length minimizers (see, for example, [5], Appendix C there). Therefore we will call them in the sequel *normal sub-Riemannian geodesics.* The following definition is a natural extension of the notion of the geodesic equivalence from Riemannian to the general sub-Riemannian case:

Definition 1 *Two sub-Riemannian metrics given on a distribution D of a manifold M are called geodesically (or projective) equivalent at a point $q_0 \in M$, if in some neighborhood of q_0 all their normal geodesics, considered as unparametrized curves, coincide.*

It is clear that if sub-Riemannian metrics G_1 and G_2 are constantly proportional, i.e., there exists a positive constant C such that $G_{2q} = CG_{2q}$ for any q, then they are geodesically equivalent. The first appearing question is whether there exist constantly non-proportional geodesically equivalent sub-Riemannian metrics? The simplest example of constantly non-proportional Riemannian metrics on a surface can be described as follows: Let P and S be a plane and a hemisphere in $\mathbb{R}^3$ such that equator of the hemisphere is parallel to the plane. Let G_1 and $\bar{G}_2$ be the metrics on P and S respectively, induced from the Euclidean metric on $\mathbb{R}^3$. Denote by $F : S \mapsto P$ the stereographic projection from the center O of the hemisphere (namely, if $q \in S$ then $F(q)$ is the only point on P lying on the straight line, which connects O and q). Then the mapping F sends geodesics of $\bar{G}_2$ (arcs of big circles on S) to geodesics of G_1 (straight lines on P). Therefore G_1 is geodesically equivalent to $G_2 = (F^{-1})^*\bar{G}_2$, the pull-back of $\bar{G}_2$ by F^{-1}, but this metric are not constantly proportional. Moreover, as E. Beltrami showed in [1], a Riemannian metric on a surface is geodesically equivalent to the flat one iff it has a constant curvature.

Let us introduced some notions, which are important for the considered problem. For a given ordered pair of sub-Riemannian metrics G_1, G_2 and a point q one can define the following linear operator $S_q : D(q) \mapsto D(q)$:

$$G_{2q}(v_1, v_2) = G_{1q}(S_q v_1, v_2), \quad v_1, v_2 \in D(q).$$

Obviously, S_q is self-adjoint w.r.t. the Euclidean structure given by G_1.

Definition 2 *The operator S_q will be called the transition operator from the metric G_1 to the metric G_2 at the point q.*

Let $N(q)$ be the number of distinct eigenvalues of the operator S_q.

Definition 3 *The point q_0 is called regular w.r.t. the pair of sub-Riemannian metrics G_1 and G_2, if the function $N(q)$ is constant in some neighborhood of q_0.*

Note that the regularity of the point q_0 is equivalent to the fact that the set of multiplicities of eigenvalues of the transition operator S_q is the same for all points q from some neighborhood of q_0 (in [7] regular points were called stable). By standard arguments one can show that the function $N(q)$ is lower semicontinuous. This together with the fact that it is integer-valued implies the following

Proposition 1 *The set of regular points w.r.t. the pair of sub-Riemannian metrics is open and dense in M.*

For Riemannian metrics on an n-dimensional manifold all possible pairs of geodesically equivalent metrics in a neighborhood of a regular point w.r.t. these metrics were described already by Levi-Civita in [4] (see Theorem 1 below and also [7]), who had extended the earlier result of Dini for surfaces (see [2],[3], or [6]) to an arbitrary n. From this result it follows that Riemannian metrics, for which there exists at least one non-proportional geodesically equivalent Riemannian metric, are of the very special form.

The classification of geodesically equivalent Riemannian metrics at non-regular points (i.e., points, where eigenvalues of transition operator bifurcate) even on a surface was not done, while the geodesic equivalence of proper sub-Riemannian metrics (i.e, when $D \neq TM$ and D is bracket-generating) was not studied before. In the present paper we treat both these problems.

In the sequel for shortness (m, n)-distribution means an m-dimensional subbundle of the tangent bundle of an n-dimensional manifold. Our study of the geodesic equivalence of proper sub-Riemannian metrics will be mainly concentrated on the following two cases:

1. D is the contact distribution. Namely, D is a corank 1 distribution on an odd dimensional manifold such that if ω is a differential 1-form, which annihilates D, $D(q) = \{v \in T_qM : \omega_q(v) = 0\}$, then the restriction $d\omega|_D$ of the differential $d\omega$ on D is a nondegenerated 2-form at any q. In this case there are no abnormal Pontryagin extremals.

2. D is the quasi-contact distribution. Namely, D is a corank 1 distribution on an even dimensional manifold such that if ω is a differential 1-form, which annihilates D, then the restriction $d\omega|_D$ of the differential $d\omega$ on D has 1-dimensional kernel at any q. The kernels of $d\omega|_D$ form line distribution. We will call it *the abnormal line distribution of the quasi-contact distribution D*. Abnormal extremal trajectories of the sub-Riemannian metric G on D are exactly the leaves of this distribution, parametrized by the length.

Clearly in both cases the germs of distribution D are generic germs of corank 1 distributions. Note also that in both cases there exists only one, up to a diffeomorphism, distribution satisfying the prescribed properties (a particular case of Darboux's Theorem). Actually, our method works for sub-Riemannian metrics defined on much more general distributions, for example, on so-called step 1 bracket-generating distributions: an (m, n)-distribution D is called step 1 bracket-generating if $\dim D^{l+1} = \dim D^l + 1$ for any $1 \leq l \leq n - m$ (here the lth power D^l of the distribution D is defined by induction $D^l = D^{l-1} + [D, D^{l-1}]$). The study of the problem of the geodesic equivalence for sub-Riemannian metrics on general step 1 bracket-generating distributions will be done in our future publications.

The paper is organized as follows. In section 2 we show that the problem of geodesic equivalence of sub-Riemannian metrics can be reduced to the question of the existence of an orbital diffeomorphism between the corresponding flows of extremals. This reduction is obvious in the Riemannian case, but in the proper sub-Riemannian case it has some additional difficulties, especially in the presence of abnormal extremals. After this reduction we express the condition for the existence of the orbital diffeomorphism in terms of the special frame adapted to the pair of sub-Riemannian metrics. Further for step 1 bracket-generating distributions we obtain a necessary condition for the geodesic equivalence in terms of divisibility of some polynomials (on the fibers of the cotangent bundle of the ambient manifold) associated with these metrics. We call it the first divisibility condition. It imposes rather strong restrictions on the pair of the metrics.

In section 3 we give the coordinate-free formulation of Levi-Civita's Theorem (Theorem 1) and prove it in a new, rather elementary way, using the conditions for the existence of the orbital diffeomorphism and the first divisibility condition. In section 4 for a sub-Riemannian metric on corank 1 distribution we obtain an additional necessary condition for the geodesic equivalence in terms of divisibility of some polynomials associated with these metrics. We call it the second divisibility condition. Using the conditions for the existence of the orbital diffeomorphism and the second divisibility condition we prove that sub-Riemannian metrics on contact distributions are geodesically equivalent iff they are constantly proportional (Theorem 2) and we give the classification of all geodesically equivalent sub-Riemannian metrics on quasi-contact distributions (Theorem 3). This classification is given in coordinate-free way and has apparent similarities with our interpretation of Levi-Civita's Theorem. This gives a hope for the existence of a general classification theorem about geodesic equivalence of sub-Riemannian metrics defined on very general class of distribution, which will contain as particular cases the cases considered in the present paper.

Finally in section 5 for the Riemannian metrics on a surface we obtain the classification of geodesically equivalent pairs at a non-regular point (the point of bifurcation of the eigenvalues of the transition operator). Note that for generic pair of Riemannian metrics on a surface the set of points of their proportionality consists of isolated points. Therefore it is natural to consider the case when two Riemannian metrics on a surface are proportional in an isolated point. Some results of the global topological nature (namely, about the number of the points of proportionality for a pair of globally geodesically equivalent Riemannian metrics on a sphere) were obtained in [6], but the local classification surprisingly was not done before. The canonical conformal structure on a surface, associated with a Riemannian metric, plays the crucial role in this classification. Using this conformal structure and Dini's Theorem (Levi-Civita's Theorem in the case of a surface), one can associate any pair of geodesically equivalent metrics on a surface, which are proportional in an isolated point q_0, with some (multiple-valued) analytic function in a neighborhood of q_0 with a singularity at q_0. The analysis of this singularity gives us the required classification (Theorems 5 and 6).

2 Geodesic equivalence and orbital diffeomorphism of the extremal flows

2.1 Existence of the orbital diffeomorphism Let G_1 and G_2 be two sub-Riemannian metrics on a distribution D of a manifold M, $||\cdot||_{1_q}$ and $||\cdot||_{2_q}$ be the corresponding Euclidean norms on $D(q)$, h_1 and h_2 be the Hamiltonians, defined by (1.1), where $||\cdot||_q$ is replaced by $||\cdot||_{i_q}$, $i=1,2$. Also let H_1 and H_2 be the $\frac{1}{2}$-level sets of h_1 and h_2 respectively, i.e.

$$H_i = \{\lambda \in T^*M : h_i(\lambda) = \frac{1}{2}\}. \tag{2.1}$$

Besides, for given distribution D and metric G on it denote by $J^k(D,G)$ the space of k-jets of all C^k curves admissible to D and parametrized by length w.r.t. the metrics G. By definition the 1-jet $J^1(D,G)$ satisfies

$$J^1(D,G) = \{(q,v) : q \in M, v \in D(q), ||v||_q = 1\}.$$

For given curve γ we will denote by $j^{(k)}_{t_0}\gamma$ the k-jet of the curve γ at the point t_0.

Proposition 2 *If for some neighborhood U of q_0 in M there exist a fiberwise diffeomorphism $\Phi : H_1 \cap T^*U \mapsto H_2 \cap T^*U$ and a function $a : H_1 \mapsto \mathbb{R}$ such that*

$$\Phi_* \vec{h}_1(\lambda) = a(\lambda)\vec{h}_2\big(\Phi(\lambda)\big), \tag{2.2}$$

then the metrics G_1 and G_2 are geodesically equivalent at q_0.

Proof. Indeed, Φ maps any trajectory of the system $\dot\lambda = \vec{h}_1(\lambda)$, lying in $H_1 \cap T^*U$, to the curve, which coincides, up to reparametrization, with a trajectory of the system $\dot\lambda = \vec{h}_2(\lambda)$. Therefore in U any normal sub-Riemannian geodesics of G_1 is, up to reparametrization, a normal sub-Riemannian geodesics of G_2. $\Box$

In the case of Riemannian metrics the relation (2.2) is also necessary for the geodesic equivalence of metrics G_1 and G_2. Indeed, in this case there is only one geodesic passing through the given point in the given direction, and the map $P_i^{(1)} : H_i \mapsto J^1(TM, G_i)$ defined by

$$P_i(\lambda) \stackrel{def}{=} j_0^1 \pi(e^{t\vec{h}_i}) = \Big(\pi(\lambda), \pi_*\big(\vec{h}_i(\lambda)\big)\Big), \quad i=1,2, \tag{2.3}$$

is a diffeomorphism (here we denote by $\pi : T^*M \mapsto M$ the canonical projection and by $e^{t\vec{h}_i}$ the flow generated by vector field $\vec{h}_i$, $i=1,2$). So, directly by definition, if the metrics G_1 and G_2 are geodesically equivalent at q_0, then there is a neighborhood U of q_0 such that the following diffeomorphism

$$\Phi(\lambda) = \left(P_2^1\right)^{-1}\left(\frac{1}{||P_1^1(\lambda)||_{2_q}} P_1^1(\lambda)\right), \quad q = \pi(\lambda), \tag{2.4}$$

is fiberwise, maps $H_1 \cap T^*U$ to $H_2 \cap T^*U$ and satisfies (2.2) on $H_1 \cap T^*U$.

Definition 4 *A fiberwise diffeomorphism Φ defined on a nonempty open set $\mathcal{B}$ of H_1 such that $\Phi(\mathcal{B}) \subset H_2$ is called the orbital diffeomorphism of the extremal flows of the sub-Riemannian metrics G_1 and G_2 on $\mathcal{B}$, if it satisfies (2.2) for any $\lambda \in \mathcal{B}$.*

Let us study the question, whether the existence of the orbital diffeomorphism is necessary for the geodesic equivalence of sub-Riemannian metrics. In the case of a proper sub-Riemannian metric (i.e., $D \neq TM$, D is bracket-generating) an entire family of normal sub-Riemannian geodesics passes in general through the given point in the given direction. So, in order to distinguish different normal geodesics, passing through a point, we need jets of higher order. Besides, the presence of the abnormal extremal trajectories causes to addition difficulties, as shown below. By analogy with (2.3) let us define the following mapping $P_i^{(k)} : H_i \mapsto J^k(D, G_i)$, $i = 1, 2$:

$$P_i^{(k)}(\lambda) \stackrel{def}{=} j_0^k \pi(e^{t\vec{h}}) \tag{2.5}$$

Then one can check without difficulties that:

a) if D is contact then the mapping $P_i^{(2)}$ establishes the diffeomorphism between H_i and its image;

b) if D is quasi-contact, $\mathcal{C}$ is the abnormal line distribution of D, and the set $\mathcal{S}_i \subset H_i$ is defined by

$$\mathcal{S}_i = \{\lambda \in H_i : P_i^1(\lambda) \in \mathcal{C}\}, \tag{2.6}$$

then the restriction of the mapping $P_i^{(2)}$ on $H_{iq} \backslash \mathcal{S}_i$ establishes the diffeomorphism between $H_{iq} \backslash \mathcal{S}_i$ and its image, while the restriction of $P_i^{(2)}$ on $\mathcal{S}_i$ is constant on each fiber.

Now denote by $\Omega_q(D, G_i)$ the set of all C^∞ admissible curves, starting at q and parametrized by length w.r.t. the metric G_i and let $J_q^k(D, G_i)$ be the space of k-jet of these curves at 0. Consider the mapping $I_q : \Omega_q(D, G_1) \mapsto \Omega_q(D, G_2)$ which sends a curve γ to its reparametrization (w.r.t. the length of G_2). Obviously , this mapping induces the diffeomorphisms $I_q^{(k)} : J_q^k(D, G_1) \mapsto J_q^k(D, G_2)$. Collecting all such diffeomorphisms for any q we obtain a diffeomorphism $I^{(k)} : J^k(D, G_1) \mapsto J^k(D, G_2)$. Then similarly to (2.4) we obtain that if the distribution D is one of the two listed in Introduction, and the sub-Riemannian metrics G_1 and G_2, defined on D, are geodesically equivalent at q_0, then there exist a neighborhood U of q_0 such that the following mapping

$$\Phi(\lambda) = \left(P_2^{(2)}\right)^{-1} \circ I^{(2)} \circ P_1^{(2)}(\lambda), \tag{2.7}$$

is well defined on the set $\mathcal{B}$, where $\mathcal{B} = H_1 \cap T^*U$ in contact case and $\mathcal{B} = (H_1 \cap T^*U) \backslash \mathcal{S}_1$ in quasi-contact (here $\mathcal{S}_1$ is as in (2.6)). Moreover, such Φ is the orbital diffeomorphism on the set $\mathcal{B}$ w.r.t. the metrics G_1 and G_2. We have proved the following

Proposition 3 *If G_1 and G_2 are Riemannian metric or sub-Riemannian metrics defined on contact or quasi-contact distributions and if they are geodesically equivalent at some point q_0, then for some neighborhood U of q_0 there exists the orbital diffeomorphism of the extremal flows of the metrics G_1 and G_2 on some nonempty open set $\mathcal{B}$ in $H_1 \cap T^*U$, $\pi(\mathcal{B}) = U$. In the Riemannian and contact case one can take $\mathcal{B} = H_1 \cap T^*U$, while in quasi-contact case one can take $\mathcal{B} = (H_1 \cap T^*U) \backslash \mathcal{S}_1$, where $\mathcal{S}_1$ is as in (2.6).*

Actually, there is an analogue of the previous proposition for sub-Riemannian metrics defined on much more wide class of distributions. To formulate it let us introduce some notations. Denote by $\mathcal{A}_{q_0}(D)$ the set of all points $q \in M$ which can be connected with q_0 by abnormal extremal trajectory of the distribution D. For example, in Riemannian and contact case $\mathcal{A}_{q_0}(D)$ is empty; in quasi-contact case $\mathcal{A}_{q_0}(D)$ is the set $L_{q_0} \backslash \{q_0\}$, where L_{q_0} is the leaf of the abnormal line distribution, passing through q_0.

Proposition 4 *Suppose that the sub-Riemannian metrics G_1 and G_2, defined on the bracket-generating distribution D, are geodesically equivalent at the point q_0 and for any neighborhood V of q_0 the set $V \backslash \mathcal{A}_{q_0}(D)$ has positive Lebesgue measure. Then for some neighborhood U of q_0 there exists the orbital diffeomorphism of the extremal flows of the metrics G_1 and G_2 on some open set $\mathcal{B}$ in $H_1 \cap T^*U$, $\pi(\mathcal{B}) = U$.*

Remark 2 *Actually in the previous proposition one can replace the set $\mathcal{A}_{q_0}(D)$ by the set of all points $q \in M$ which can be connected by abnormal extremal trajectory, having minimal length w.r.t. the metric G_1 (or G_2) among all admissible trajectories with endpoints q_0 and q.*

Since in the present paper we solve completely the problem of geodesic equivalence only in the cases, considered in Proposition 3, we postpone the proof of Proposition 4 and the statement in Remark 2 to the future paper.

2.2 The orbital diffeomorphism in terms of the adapted frame to the pair of metrics. Suppose that D is an (m,n)-distribution on a manifold M. Let q_0 be a regular point w.r.t. the metric G_1 and G_2 (see Definition 3). It is simple to show that the regularity of the point q_0 is equivalent to the fact that the set of the multiplicities of the eigenvalues of the transition operator S_q is the same for all points q from some neighborhood of q_0. Therefore in some neighborhood U of q_0 one can choose the basis $(X_1,\ldots,X_m)$ of the distribution D orthonormal w.r.t. the metric G_1 such that each $X_i(q)$ is eigenvector of the transition operator S_q, $q\in V$. Such basis of D will be called the *adapted basis to the ordered pair of metrics* (G_1,G_2) *on a set* U. A frame $(X_1,\ldots,X_n)$ will be called the *adapted frame to the ordered pair of sub-Riemannian metrics* (G_1,G_2) *on a set* U, if the tuple $\bigl(X_1,\ldots,X_m\bigr)$ is the adapted basis of D w.r.t. (G_1,G_2) on U.

Let us express the relation (2.2) for the orbital diffeomorphism in terms of some adapted frame $(X_1,\ldots,X_n)$. Let $u_i: T^*M\mapsto\mathbb{R}$ be the "quasi-impulse" of the vector field X_i,

$$u_i(p,q)=p\bigl(X_i(q)\bigr),\quad q\in U, p\in T^*U. \tag{2.8}$$

For given diffeomorphism Φ defined on an open set of T^*M denote by

$$\Phi_i=u_i\circ\Phi,\quad 1\le i\le n. \tag{2.9}$$

Suppose also that for any i, $1\le i\le m$, the eigenvalue of the transition operator S_q, corresponding to the eigenvector $X_i(q)$, is equal to $\alpha_i^2(q)$.

Lemma 1 *If Φ is the orbital diffeomorphism of the extremal flows of the metrics G_1 and G_2 on an open set $\mathcal{B}\subset H_1\cap T^*U$, then the functions Φ_i with $1\le i\le m$ satisfy*

$$\Phi_i=\frac{\alpha_i^2u_i}{\sqrt{\sum_{k=1}^m\alpha_k^2u_k^2}},\quad 1\le i\le m. \tag{2.10}$$

Proof. Since by construction the tuple $(X_1,\ldots,X_m)$ constitute an orthonormal basis of the distribution D w.r.t. the metric G_1, the Hamiltonian h_1 satisfies $h_1=\frac{1}{2}\sum_{i=1}^m u_i^2$, and

$$\vec h_1=\sum_{i=1}^m u_i\vec u_i,\quad \pi_*\vec h_1=\sum_{i=1}^m u_iX_i,\quad H_1=\Bigl\{\lambda\in T^*U:\sum_{i=1}^m u_i^2=1\Bigr\} \tag{2.11}$$

(here $\pi: T^*M\mapsto M$ is the canonical projection). Let $\bar X_i$ be

$$\bar X_i=\frac{1}{\alpha_i}X_i,\quad 1\le i\le m, \tag{2.12}$$

and $\bar u_i(p,q)=p\bigl(\bar X_i(q)\bigr)$ be the corresponding quasi-impulses. Then

$$\bar u_i=\frac{u_i}{\alpha_i},\quad 1\le i\le m, \tag{2.13}$$

Note that by construction $(\bar X_1,\ldots,\bar X_n)$ is the orthonormal basis of D w.r.t. the metric G_2. Hence, similarly to (2.11), we have $\vec h_2=\sum_{i=1}^m\bar u_i\vec{\bar u}_i$, which together with (2.12) and (2.13) implies that

$$\pi_*\vec h_2=\sum_{i=1}^m\frac{u_i}{\alpha_i^2}X_i,\quad H_2=\Bigl\{\lambda\in T^*U:\sum_{i=1}^m\bar u_i^2=1\Bigr\}=\Bigl\{\lambda\in T^*U:\sum_{i=1}^m\frac{u_i^2}{\alpha_i^2}=1\Bigr\} \tag{2.14}$$

Suppose that Φ is the orbital diffeomorphism on some set $\mathcal{B}$, satisfying (2.2) for some function a. Then by definition $\Phi(\lambda) \in H_2$ for any $\lambda \in \mathcal{B}$. This together with (2.9) and (2.14) implies that

$$\sum_{i=1}^{m} \frac{\Phi_i^2}{\alpha_i^2} = 1. \tag{2.15}$$

Further from the fact that Φ is fiberwise and (2.11) it follows that

$$(\pi_* \circ \Phi_*)\vec{h}_1(\lambda) = \pi_* \vec{h}_1(\lambda) = \sum_{i=1}^{m} u_i X_i.$$

On the other hand, (2.9) and (2.14) imply

$$\pi_* \vec{h}_2\big(\Phi(\lambda)\big) = \sum_{i=1}^{m} \frac{\Phi_i}{\alpha_i^2} X_i.$$

From the last two relations and (2.2) it follows that

$$a\Phi_i = \alpha_i^2 u_i, \quad 1 \leq i \leq m$$

From this and (2.15) it follows easily that

$$a = \sqrt{\sum_{k=1}^{m} \alpha_k^2 u_k^2}, \tag{2.16}$$

which implies (2.10). $\Box$

Now we will find the relation for the remaining components Φ_i, $m+1 \leq i \leq n$, of Φ. Let c_{ji}^k be the structural functions of the adapted frame $(X_1, \ldots, X_n)$, i.e., the function, satisfying $[X_i, X_j] = \sum c_{ji}^k X_k$. Let the vector fields X_i, $1 \leq i \leq m$, satisfy (2.12) and set

$$\bar{X}_i = X_i, \quad m+1 \leq i \leq n. \tag{2.17}$$

Note that by construction $(\bar{X}_1, \ldots, \bar{X}_n)$ is the adapted frame w.r.t. the ordered pair (G_2, G_1). Let $\bar{c}_{ji}^k$ be the structural functions of the frame $(\bar{X}_1, \ldots, \bar{X}_n)$. The following functions will be very useful in the sequel together with function a, defined by (2.16):

$$R_j \overset{def}{=} \frac{1}{2}\vec{h}_1(\alpha_j^2)u_j + \alpha_j^2 \vec{h}_1(u_j) - \frac{1}{2}\alpha_j^2 u_j \frac{\vec{h}_1(a^2)}{a^2} - \sum_{1 \leq i,k \leq m} \bar{c}_{ji}^k \alpha_i \alpha_j \alpha_k u_i u_k, \tag{2.18}$$

$$Q_{jk} \overset{def}{=} \sum_{i=1}^{m} \bar{c}_{ji}^k \alpha_i u_i \tag{2.19}$$

Lemma 2 *A map Φ is the orbital diffeomorphism on a set $\mathcal{B}$ of the extremal flows of the metrics G_1 and G_2 iff on $\mathcal{B}$ the functions Φ_k with $m+1 \leq k \leq n$ satisfy the following relations:*

$$\forall 1 \leq j \leq m: \quad \alpha_j \sum_{k=m+1}^{n} Q_{jk}\Phi_k = \frac{R_j}{a} \tag{2.20}$$

$$\forall m+1 \leq s \leq n: \quad \vec{h}_1(\Phi_s) - \sum_{k=m+1}^{n} Q_{sk}\Phi_k = \frac{1}{a}\sum_{k=1}^{m} Q_{sk}\alpha_k u_k. \tag{2.21}$$

Proof. In the sequel we set

$$\forall m+1 \leq n: \quad \alpha_i \equiv 1. \tag{2.22}$$

Denote by Y_i the vector field on H_1, which is the lift of the vector field X_i (i.e., $\pi_* Y_i = X_i$), and $du_j(Y_i) = 0$ $\forall 1 \leq j \leq n$ (i.e., Y_j is horizontal field of the connection on T^*M defined by distribution, satisfying $du_1 = \ldots = du_n = 0$). Similarly, let $\bar{Y}_i$ be the vector field on H_2, which is the lift of $\bar{X}_i$ and $d\bar{u}_j(Y_i) = 0$ for all $1 \leq j \leq n$. Note also that the tuple $(u_1, \ldots, u_n)$ defines the coordinates on each fiber T_q^*M of T^*M. So, one can define the vector fields ∂_{u_i}, $1 \leq i \leq n$, as follows: ∂_{u_i} is vertical (i.e., tangent to the fibers of T^*M) and $du_j(\partial u_i) = \delta_{ij}$ for all $j = 1, \ldots n$, where δ_{ij} is the Kronecker symbol. In the same way one can define the fields $\partial_{\bar{u}_i}$. With this notations, using (2.12) and (2.13), $\forall 1 \leq i \leq n$ one can easily obtain the following relation :

$$\partial_{\bar{u}_i} = \alpha_i \partial_{u_i}, \quad \bar{Y}_i = \frac{1}{\alpha_i}\left(Y_i + \sum_{j=1}^{m} \frac{X_j(\alpha_i)}{\alpha_i} u_j \partial_{u_j}\right) \tag{2.23}$$

Besides, by standard calculations, we have

$$\vec{h}_1 = \sum_{i=1}^{m} u_i \vec{u}_i = \sum_{i=1}^{m} u_i Y_i + \sum_{i=1}^{m} \sum_{j,k=1}^{n} c_{ji}^k u_i u_k \partial_{u_j}, \tag{2.24}$$

$$\vec{h}_2 = \sum_{i=1}^{m} \bar{u}_i \vec{\bar{u}}_i = \sum_{i=1}^{m} \bar{u}_i \bar{Y}_i + \sum_{i=1}^{m} \sum_{j,k=n}^{m} \bar{c}_{ji}^k \bar{u}_i \bar{u}_k \partial_{\bar{u}_j}. \tag{2.25}$$

Substituting (2.13) and (2.23) into (2.25), we obtain

$$\vec{h}_2 = \sum_{i=1}^{m} \frac{u_i}{\alpha_i^2} Y_i + \sum_{i,j=1}^{m} \frac{X_i(\alpha_j)}{\alpha_i^2 \alpha_j} u_i u_j \partial_{u_j} + \sum_{i=1}^{m} \sum_{j,k=1}^{n} \frac{\bar{c}_{ji}^k \alpha_j}{\alpha_i \alpha_k} u_i u_k \partial_{u_j}. \tag{2.26}$$

This together with (2.10) implies easily that

$$\vec{h}_2(\Phi(\lambda)) = a^{-1} \sum_{i=1}^{m} u_i Y_i + a^{-2} \sum_{j=1}^{m} \Big(\frac{1}{2} \vec{h}_1(\alpha_j^2) u_j + \sum_{i,k=1}^{m} \bar{c}_{ji}^k \alpha_i \alpha_j \alpha_k u_i u_k \Big) \partial_{u_j} +$$

$$a^{-1} \sum_{j=1}^{m} \sum_{k=m+1}^{n} \sum_{i=1}^{m} \bar{c}_{ji}^k \alpha_i \alpha_j u_i \Phi_k \partial_{u_j} + \sum_{s=m+1}^{n} \sum_{i=1}^{m} \Big(a^{-2} \sum_{k=1}^{m} \bar{c}_{si}^k \alpha_i \alpha_k u_i u_k + \tag{2.27}$$

$$a^{-1} \sum_{k=m+1}^{n} \bar{c}_{si}^k \alpha_i u_i \Phi_k \Big) \partial_{u_s}, \tag{2.28}$$

where a is as in (2.16). On the other hand, from the fact that Φ is fiberwise and relation (2.10) it follows that

$$\Phi_* \vec{h}_1(\lambda) = \sum_{i=1}^{m} u_i Y_i + \sum_{j=1}^{m} \vec{h}\left(\frac{\alpha_j^2 u_j}{\sqrt{\sum_{l=1}^{m} \alpha_l^2 u_l^2}} \right) \partial_{u_j} + \sum_{j=m+1}^{n} \vec{h}(\Phi_j) \partial_{u_j}. \tag{2.29}$$

Using relations (2.27) and (2.29), it is not hard to check by direct calculations that (2.2) holds iff both (2.20) and (2.21) hold, which concludes the proof of the Lemma. □

2.3 The first divisibility condition. Let $\mathcal{I}_1 : D(q)^* \mapsto D(q)$ be the canonical isomorphism w.r.t. the inner product $G_{1q}(\cdot,\cdot)$, namely, $\ell(\cdot) = G_{1q}(\mathcal{I}_1(\ell),\cdot)\ \forall \ell \in D(q)^*$. Define the following function $\mathcal{P} : T^*M \mapsto \mathbb{R}$:

$$\mathcal{P}(p,q) = \Big(||\mathcal{I}_1(p|_{D(q)})||_{2_q}\Big)^2, \quad q \in M, p \in T_q^*M \tag{2.30}$$

(here $||\cdot||_{2_q}$ is the Euclidean norm on $D(q)$ corresponding to the inner product $G_{2q}(\cdot,\cdot)$, $p|_{D(q)}$ is the restriction of covector $p \in T_q^*M$ on the subspace $D(q)$). Obviously, the restriction of $\mathcal{P}$ on each fiber T_q^*M is a degree 2 homogeneous polynomial, while the restriction of $\vec{h}_1(\mathcal{P})$ on each fiber T_q^*M is a degree 3 polynomial. Besides, in a neighborhood of the regular point

$$\mathcal{P} = a^2 = \sum_{i=1}^{m} \alpha_i^2 u_i^2, \tag{2.31}$$

where $(u_1,\ldots,u_m)$ are quasi-impulses of the vectors of the adapted basis $(X_1,\ldots X_m)$ to the order pair (G_1,G_2) and α_i^2 are eigenvalues of the transition operator S_q, corresponding to the eigenvectors X_i.

Definition 5 *We will say that the ordered pair (G_1,G_2) of sub-Riemannian metrics on the distribution D satisfies the first divisibility condition on a set U, if the polynomial $\vec{h}_1(\mathcal{P})|_{T_q^*M}$ is divided by the polynomial $\mathcal{P}|_{T_q^*M}$ for any $q \in U$.*

Proposition 5 *Let D be an (m,n)-distribution on a manifold M such that*

$$\forall 1 \le s \le n-m+1, \quad \dim D^s = m+s-1. \tag{2.32}$$

Suppose also that for given two sub-Riemannian metrics G_1 and G_2 on D and for some open set U of M there exists an orbital diffeomorphism of the extremal flows of these metrics in some open set $\mathcal{B}$ in $H_1 \cap T^*U$, $\pi(\mathcal{B}) = U$. Then the pair (G_1,G_2) satisfies the first divisibility condition on U.

Proof. Since the set of regular points is dense (Proposition 1) it is sufficient to prove the first divisibility condition for a regular point q_0 w.r.t. the pair (G_1,G_2). Therefore in order to obtain the first divisibility condition we can use Lemmas 1 and 2. Note also that by (2.18) the function R_j has the following form on each fiber:

$$R_j = -\frac{1}{2}\alpha_j^2 u_j \frac{\vec{h}_1(\mathcal{P})}{\mathcal{P}} + \text{polynomial} \tag{2.33}$$

First suppose that $D = TM$ (in this case the assumption (2.32) holds automatically). Then the identity (2.20) is equivalent to the identity $R_j \equiv 0$, $1 \le j \le n$, which holds on open set $\mathcal{B}$ in $H_1 \cap T^*U$ with $\pi(\mathcal{B}) = U$ and therefore on the whole T^*U. Hence from (2.33) it follows that $u_j \frac{h_1(\mathcal{P})}{\mathcal{P}}$, has to be a polynomial, which implies easily that the polynomial $\mathcal{P}$ has to divide the polynomial $\vec{h}_1(\mathcal{P})$, i.e., the first divisibility condition holds.

Now consider the case $D \ne TM$. By assumption (2.32), we can complete the adapted basis $(X_1,\ldots,X_m)$ to the adapted frame such that

$$\forall m+1 \le s \le n \quad X_s \in D^{s-m+1} \tag{2.34}$$

Then $D^2 = \mathrm{span}(X_1, \dots X_{m+1})$, which implies that there exist indices $\bar{i}, \bar{j}$, $1 \le \bar{i}, \bar{j} \le m$, such that $\bar{c}_{\bar{j}\bar{i}}^{m+1}(q_0) \ne 0$, while $\bar{c}_{ij}^k = 0$ for all $1 \le i, j \le m$ and $k > m+1$. In other words,

$$\forall k, j : \, k > m+1, 1 \le j \le m \quad Q_{jk} \equiv 0 \tag{2.35}$$

$$\exists \bar{j} : 1 \le \bar{j} \le m, \quad Q_{\bar{j}m+1} \not\equiv 0 \tag{2.36}$$

(see (2.19) for the definition of the functions Q_{jk}). Then from (2.20) it follows that

$$\Phi_{m+1} = \frac{R_{\bar{j}}}{\alpha_{\bar{j}} Q_{\bar{j}m+1} \sqrt{\mathcal{P}}}. \tag{2.37}$$

Using (2.33), we obtain

$$\Phi_{m+1} = -\frac{1}{2} \alpha_{\bar{j}} u_{\bar{j}} \frac{\vec{h}_1(\mathcal{P})}{Q_{\bar{j}m+1} \mathcal{P}^{3/2}} + \frac{1}{\alpha_{\bar{j}} Q_{\bar{j}m+1} \sqrt{\mathcal{P}}} \text{ polynomial} \tag{2.38}$$

on each set $\mathcal{B} \cap T_q^*M$, $q \in U$.

Further, from assumption (2.32) it follows that $\bar{c}_{si}^{k+1} = 0$ for any k, s, i such that $m < s < k$ and $1 \le i \le m$. On the other hand, there exist $\bar{i}$, $1 \le \bar{i} \le m$, such that $\bar{c}_{s\bar{i}}^{s+1} \ne 0$. In other words,

$$\forall k, \, s : m < s < k \quad Q_{s\,k+1} \equiv 0 \tag{2.39}$$

$$\forall s : m < s < n-1 \quad Q_{s\,s+1} \not\equiv 0. \tag{2.40}$$

Hence by (2.21), applied for $s = m+l-1$ with $2 \le l \le n-m$, one has

$$\Phi_{m+l} = Q_{m+l-1\,m+l}^{-1} \Big(\vec{h}_1(\Phi_{m+l-1}) - \sum_{k=m+1}^{m+l-1} Q_{m+l-1\,k} \Phi_k - \mathcal{P}^{-1/2} \sum_{k=1}^{m} Q_{m+l-1,k} \alpha_k u_k \Big). \tag{2.41}$$

Then by induction from (2.38) and (2.41) it is not difficult to get the following relation for any $2 \le l \le m-n$

$$\Phi_{m+l} = \frac{(-1)^l (2l-1)!! \, \alpha_{\bar{j}} u_{\bar{j}} \big(\vec{h}_1(\mathcal{P})\big)^l}{2^l Q_{\bar{j}\,m+1} \mathcal{P}^{l+1/2} \prod_{i=1}^{l-1} Q_{m+i\,m+i+1}} + \frac{\text{polynomial}}{Q_{\bar{j}\,m+1}^l \mathcal{P}^{l-1/2} \prod_{i=1}^{l-1} Q_{m+i\,m+i+1}^{l-i}} \tag{2.42}$$

on each $\mathcal{B} \cap T_q^*M$, $q \in U$. Substituting the expression for Φ_{m+l} from (2.42) to identity (2.21) with $s = n$ one can obtain without difficulties that

$$\frac{u_{\bar{j}} \big(\vec{h}_1(\mathcal{P})\big)^{n-m+1}}{Q_{\bar{j}\,m+1} \mathcal{P}^{n-m+3/2} \prod_{i=1}^{n-m-1} Q_{m+i\,m+i+1}} = \frac{\text{polynomial}}{Q_{\bar{j}\,m+1}^{n-m+1} \mathcal{P}^{n-m+1/2} \prod_{i=1}^{n-m-1} Q_{m+i\,m+i+1}^{n-m-i+1}} \tag{2.43}$$

or, equivalently

$$\frac{u_{\bar{j}} Q_{\bar{j}\,m+1}^{n-m} \big(\prod_{i=1}^{n-m-1} Q_{m+i\,m+i+1}^{n-m-i}\big) \big(\vec{h}_1(\mathcal{P})\big)^{n-m+1}}{\mathcal{P}} = \text{polynomial}. \tag{2.44}$$

on each set $\mathcal{B} \cap T_q^*M$, $q \in U$. Note that the left-hand side of (2.44) is rational function. Hence from (2.44) it has to be polynomial. Note also that $\mathcal{P}$ is positive definite quadratic form (see (2.31)), while the functions Q_{jk} are linear (with real coefficients) on each fiber. Therefore from (2.44) it follows easily that the polynomial $\mathcal{P}$ has to divide the polynomial $\vec{h}_1(\mathcal{P})$, i.e., the first divisibility condition holds. The proof of the proposition is concluded. $\square$.

Note that if D is contact, quasi-contact, or $D = TM$, then the assumption (2.32) of the previous proposition holds. So, as a direct consequence of Proposition 3 and the previous proposition, we have the following

Corollary 1 *Suppose that two metrics G_1 and G_2 defined on the distribution D are geodesically equivalent at the point q_0. Assume also that the distribution D satisfies one of the two following conditions:*

1. *$D = TM$ (the Riemannian case);*

2. *D is corank 1 contact or quasi-contact distribution;*

Then the pair (G_1, G_2) satisfies the first divisibility condition on U.

So, in the cases under consideration the first divisibility condition is necessary for the geodesic equivalence. In the next proposition we collect all information from the first divisibility condition, which will be used in the sequel. It shows that the first divisibility condition imposes rather strong restrictions on the pair of the metrics.

Proposition 6 *Suppose that the metrics G_1 and G_2, defined on the distribution D, satisfy the first divisibility condition on some set U. If $(X_1, \dots X_m)$ is a basis of D adapted to the order pair (G_1, G_2), and the transition operator S_q has the form $S_q = \operatorname{diag}(\alpha_1^2(q), \dots, \alpha_m^2(q))$ in this basis ($\alpha_i > 0$), then the following relations hold*

$$[X_i, X_j](q) \notin D(q) \;\Rightarrow\; \alpha_i(q) = \alpha_j(q); \tag{2.45}$$

$$X_i\left(\frac{\alpha_j^2}{\alpha_i^2}\right) = 2c_{ji}^j\left(1 - \frac{\alpha_j^2}{\alpha_i^2}\right); \tag{2.46}$$

$$X_i\left(\frac{\alpha_j^2}{\alpha_i}\right) = 0, \quad \alpha_i \neq \alpha_j \tag{2.47}$$

$$X_i\left(\frac{\alpha_j}{\alpha_k}\right) = 0, \quad \alpha_j \neq \alpha_i, \alpha_k \neq \alpha_i; \tag{2.48}$$

$$(\alpha_j^2 - \alpha_i^2)c_{ji}^k + (\alpha_j^2 - \alpha_k^2)c_{jk}^i + (\alpha_i^2 - \alpha_k^2)c_{ik}^j = 0, \quad i, j, k \text{ are pairwise distinct}. \tag{2.49}$$

(in all relations above $1 \leq i, j, k \leq m$).

Proof. As before let us complete the adapted basis $(X_1, \dots X_m)$ of D somehow to the local frame. From (2.24) and (2.31) by direct calculation one has

$$\vec{h}_1(\mathcal{P}) = \sum_{i,j=1}^{m} X_i(\alpha_j^2)u_i u_j^2 + 2\sum_{i,j=1}^{m}\sum_{k=1}^{n} c_{ji}^k \alpha_j^2 u_i u_j u_k \tag{2.50}$$

On the other hand by the first divisibility condition there exist functions $p_i(q)$, $1 \leq i \leq n$, such that

$$\vec{h}_1(\mathcal{P}) = \Big(\sum_{i=1}^{n} p_i u_i\Big)\Big(\sum_{j=1}^{m} \alpha_j^2 u_j^2\Big). \tag{2.51}$$

Relation (2.49) follows immediately from comparing the coefficients of $u_i u_j u_k$ in the right-hand sides of (2.50) and (2.51), where i, j, k are pairwise distinct and $1 \leq i, j, k \leq m$.

Further, comparing the coefficient of $u_i u_j u_k$ in the right-hand side of (2.50)and (2.51), where $1 \leq i \leq j \leq m$ and $k > m$, we have

$$c_{ji}^k(\alpha_i^2 - \alpha_j^2) = 0 \tag{2.52}$$

Therefore, if $[X_i, X_j](q) \notin D(q)$, then there exists $k > m$ such that $c_{ji}^k(q) \neq 0$, which implies that $\alpha_i(q) = \alpha_j(q)$. Relation (2.45) is proved.

Further, comparing coefficients of u_i^3 in the right-hand sides of (2.50) and (2.51) we obtain that

$$p_i = \frac{X_i(\alpha_i^2)}{\alpha_i^2}, \tag{2.53}$$

while comparing coefficients of $u_i u_j^2$ with $i \neq j$ and using (2.53) one obtains easily that

$$\alpha_i^2 X_i(\alpha_j^2) - \alpha_j^2 X_i(\alpha_i^2) = 2c_{ji}^j(\alpha_i^2 - \alpha_j^2)\alpha_i^2. \tag{2.54}$$

The last equation implies (2.46).

In order to prove (2.47) note that we can obtain one more relation in addition to (2.46), starting with the metric G_2 as the original one and using transition from the metric G_2 to the metric G_1. Namely, if $\bar{X}_i$ is as in (2.12) then by analogy with (2.54) we have

$$\bar{\alpha}_i^2 \bar{X}_i(\bar{\alpha}_j^2) - \bar{\alpha}_j^2 \bar{X}_i(\bar{\alpha}_i^2) = 2\bar{c}_{ji}^j(\bar{\alpha}_i^2 - \bar{\alpha}_j^2)\bar{\alpha}_i^2. \tag{2.55}$$

Obviously, $\bar{\alpha}_i = \frac{1}{\alpha_i}$. Also, by (2.12) one get easily that

$$\bar{c}_{ji}^j = \frac{c_{ji}^j}{\alpha_i} - \frac{X_i(\alpha_j)}{\alpha_i \alpha_j}. \tag{2.56}$$

Substituting the last two relations and (2.12) into (2.55) it is not difficult to get the following

$$\alpha_i^2 X_i(\alpha_j^2) - \alpha_j^2 X_i(\alpha_i^2) = 2\alpha_j\big((\alpha_j c_{ji}^j - X_i(\alpha_j)\big)(\alpha_i^2 - \alpha_j^2) \tag{2.57}$$

Then combining (2.54) with (2.57) and using the fact that $\alpha_i \neq \alpha_j$ one get

$$c_{ji}^j = \frac{1}{2}\frac{X_i(\alpha_j^2)}{\alpha_j^2 - \alpha_i^2} \tag{2.58}$$

Substituting the last relation again in (2.54) we have

$$2\alpha_i^2 X_i(\alpha_j^2) - \alpha_j^2 X_i(\alpha_i^2) = 0,$$

which is equivalent to (2.47). Relation (2.48) follows immediately from (2.47).

Corollary 2 *If D is a bracket-generating $(2, n)$-distribution, $n > 2$, and the metrics G_1 and G_2, defined on D, satisfy the first divisibility condition, then they are proportional, namely $G_{2_q} = \alpha(q) G_{1_q}$.*

Proof. Since D is bracket-generating, the set V_1 of points q with

$$\dim D^2(q) = 3 \tag{2.59}$$

is open and dense. By Proposition 1 the intersection V_2 of this set with the set of all regular points w.r.t. the metrics G_1 and G_2 is also open dense. Therefore it is sufficient to proof the corollary for the points of V_2. From regularity it follows the existence of the adapted frame (X_1, X_2). So, we can apply the previous proposition. By (2.59), $[X_1, X_2] \notin D$. Hence from (2.45) it follows that $\alpha_1 \equiv \alpha_2$, which completes the proof of the corollary. □

Suppose that D is a step 1 bracket-generating $(2, n)$-distribution ($\dim D^{l+1} = \dim D^l + 1$ for any $1 \leq l \leq n - m$). It can be shown easily that D satisfy the assumptions of Proposition 4. Therefore by Propositions 4, 5, and Corollary 2 we have the following

Proposition 7 *Suppose that two sub-Riemannian metrics G_1 and G_2 are defined on a step 1 bracket-generating $(2,n)$-distribution, where $n > 2$. If they are geodesically equivalent at some point q_0, then they are proportional, namely $G_{2_q} = \alpha(q) G_{1_q}$ in some neighborhood of q_0.*

Our conjecture is that the factor $\alpha(q)$ in the previous propoposition has to be constant, but we can prove it still only in the case $n = 3$ (see Corollary 5 below). We finish this section with the following useful lemma

Lemma 3 *Suppose that the metrics G_1 and G_2, defined on an (m,n)-distribution D, satisfy the first divisibility condition at some neighborhood U of a regular point q_0. If $(X_1, \dots X_m)$ is a basis of D adapted to the order pair (G_1, G_2), and the transition operator S_q has the form $S_q = \mathrm{diag}\,(\alpha_1^2(q), \dots, \alpha_m^2(q))$ in this basis ($\alpha_i > 0$), then the functions R_j, $1 \le j \le m$, defined by (2.18), can be written in the following form*

$$R_j = \sum_{i=1}^{m}(1-\delta_{ji})\Big((\alpha_j^2 - \alpha_i^2)c_{ji}^i - \frac{X_j(\alpha_i^2)}{2}\Big)u_i^2 + \sum_{i=1}^{m}(1-\delta_{ji})\frac{\alpha_i^2}{2\alpha_j^2}X_i\Big(\frac{\alpha_j^4}{\alpha_i^2}\Big)u_i u_j +$$
$$\sum_{i=1}^{m}\sum_{k=1}^{m}(1-\delta_{ik})(\alpha_j^2-\alpha_k^2)c_{ji}^k u_i u_k + \alpha_j^2 \sum_{i=1}^{m}\sum_{k=m+1}^{n} c_{ji}^k u_i u_k. \tag{2.60}$$

(here δ_{ij} is the Kronecker symbol).

The relation can be obtain without difficulties by substitution of (2.53), (2.56) and the following obvious identity

$$\bar{c}_{i,j}^k = c_{i,j}^k \frac{\alpha_k}{\alpha_i \alpha_j} \quad i, j, k \text{ are pairwise distinct} \tag{2.61}$$

into (2.18).

3 The case of Riemannian metrics near regular point

In the present section, using the technique developed above, we give a new proof of classical Levi-Civita's Theorem about the classification of all Riemannian geodesically equivalent metrics in a neighborhood of the regular points w.r.t. these metrics (see [4], [7]). This proof is rather elementary and transparent from the geometrical point of view. Some crucial ideas of this proof will be used in the next section for obtaining the corresponding classification for sub-Riemannian geodesically equivalent metrics on quasi-contact distributions.

Here we prefer the coordinate-free formulation of Levi-Civita's Theorem, which in our opinion clarifies the statement of it. But before let us introduce some notations and prove some preparatory lemmas.

Let G_1 and G_2 be Riemannian metrics on an n-dimensional manifold M. Let q_0 be a regular point w.r.t. these metrics. Suppose that $(X_1, \dots X_n)$ is a frame adapted to the order pair (G_1, G_2) in some neighborhood of q_0, and the transition operator S_q from the metric G_1 to the metric G_2 has the form $S_q = \mathrm{diag}\,(\alpha_1^2(q), \dots, \alpha_n^2(q))$ in this basis ($\alpha_i > 0$).

Let R_j, $1 \le j \le n$, be as in (2.18). Propositions 2, 3 and Lemma 2 imply the following

Lemma 4 *Two Riemannian metrics G_1 and G_2 are geodesically equivalent at a regular point q_0 if and only if there exist some neighborhood U of q_0 such that the following identities hold on T^*U*

$$\forall j : 1 \le j \le n \quad R_j \equiv 0. \tag{3.1}$$

Further, let $\{\lambda_1, \ldots, \lambda_N\}$ be the set of all distinct eigenvalues of S_q, $\lambda_s > 0$ (from the regularity the number of these eigenvalues is constant for all q from some neighborhood of q_0). Denote by

$$I_s = \{i : \alpha_i^2 = \lambda_s\} \quad 1 \le s \le N \tag{3.2}$$

Denote also by D_s the following rank $|I_s|$-distribution

$$D_s = \text{span}\{X_i\}_{i \in I_s}, \quad 1 \le s \le N \tag{3.3}$$

Lemma 5 *If two Riemannian metrics G_1 and G_2 are geodesically equivalent at a regular point q_0, then the distribution D_s is integrable in a neighborhood of q_0.*

Proof. By Lemma 4 identities (3.1) hold. Taking the coefficient of $u_i u_k$ from (2.60), by (3.1) one has

$$(\alpha_j^2 - \alpha_k^2) c_{ji}^k + (\alpha_j^2 - \alpha_i^2) c_{jk}^i = 0, \quad i, j, k \text{ are pairwise distinct.} \tag{3.4}$$

If $\alpha_i = \alpha_j$ and $\alpha_i \ne \alpha_k$, then from the last relation $(\alpha_j^2 - \alpha_k^2) c_{ji}^k = 0$, which implies that $c_{ji}^k = 0$. In other words, if $i, j \in I_s$, then $[X_i, X_j] \in D_s$. So, D_s is an integrable distribution. □

Lemma 6 *If two Riemannian metrics G_1 and G_2 are geodesically equivalent at a regular point q_0, then the distribution*

$$D_{s,l} \stackrel{def}{=} \text{span}\big(D_s, D_l\big) = \text{span}\Big\{X_i\Big\}_{i \in (I_s \cup I_l)},$$

is integrable in a neighborhood of q_0 for all s, l , $1 \le s \ne l \le N$.

Proof. By the previous lemma it is sufficient to prove that for any three indices i, j, k with pairwise distinct α_i, α_j, and α_k we have $c_{ji}^k \ne 0$. Making the corresponding permutation of indices in (3.4), we obtain one more relation

$$-(\alpha_j^2 - \alpha_k^2) c_{ji}^k + (\alpha_i^2 - \alpha_j^2) c_{ik}^j = 0, \quad i, j, k \text{ are pairwise distinct.} \tag{3.5}$$

Combining (3.4), (3.5), and (2.49), we obtain the system of three linear equations w.r.t. c_{ji}^k, c_{jk}^i, and c_{ik}^j with the determinant equal to $2(\alpha_i^2 - \alpha_j^2)(\alpha_i^2 - \alpha_k^2)(\alpha_k^2 - \alpha_j^2)$, which implies that $c_{ji}^k = 0$. □

From the previous lemma by standard arguments one has the following

Corollary 3 *If two Riemannian metrics G_1 and G_2 are geodesically equivalent at a regular point q_0, then in some neighborhood U of the point q_0 there exist coordinates $(x_1, \ldots x_n)$ such that*

$$\forall s : 1 \le s \le p \quad D_s = \{d\, x_i = 0\}_{i \notin I_s}. \tag{3.6}$$

In other words, in this coordinates the leaves of the integrable distribution D_s are $|I_s|$-dimensional linear subspaces, parallel to the coordinate $\{x_i\}_{i \in I_s}$-subspace.

For any s, $1 \le s \le N$, denote by $\mathcal{F}_s$ the foliation of the integral manifolds of the distribution D_s. Let $\mathcal{F}_s(q_0)$ be the leaf of $\mathcal{F}_s$, passing through the point q_0. Also, let U be the neighborhood of q_0 from Corollary 3. Then for any s, $1 \le s \le N$, one can define a special map $pr_s : U \mapsto \mathcal{F}_s(q_0)$ in the following way: the point $pr_s(q)$ is the point of intersection of $\mathcal{F}_s(q_0)$ with the integral manifold of the distribution $\text{span}\{D_l : 1 \le l \le N, l \ne s\}$, passing through q. In the coordinates of Corollary 3 with $q_0 = (0, \ldots, 0)$ the map pr_s is the projection on the coordinate $\{x_i\}_{i \in I_s}$-subspace (which preserves all coordinates x_i, $i \in I_s$). Now we are ready to formulate Levi-Civita's Theorem:

Theorem 1 *(Levi-Civita) Two Riemannian metrics G_1 and G_2 are geodesically equivalent at a point q_0 if and only if for any s, $1 \leq s \leq N$, on a manifold $\mathcal{F}_s(q_0)$ there exist a Riemannian metric g_s and a positive function β_s, which is constant if* $\dim\, \mathcal{F}_s > 1$*, such that $\beta_s(q_0) \neq \beta_l(q_0)$ for all $s \neq l$ and in some neighborhood of q_0 the metrics G_1 and G_2 have the following form*

$$G_1 = \sum_{s=1}^{N} \gamma_s (pr_s)^* g_s, \tag{3.7}$$

$$G_2 = \sum_{s=1}^{N} \lambda_s \gamma_s (pr_s)^* g_s, \tag{3.8}$$

where

$$\lambda_s = (\beta_s \circ pr_s) \prod_{l=1}^{N} (\beta_l \circ pr_l), \tag{3.9}$$

$$\gamma_s = \prod_{l \neq s} \left| \frac{1}{(\beta_l \circ pr_l)} - \frac{1}{(\beta_s \circ pr_s)} \right|. \tag{3.10}$$

Proof. We start with the proof of the "only if" part. Below we work in the coordinate neighborhood U of Corollary 3. First let us prove the following

Lemma 7 *For any s, $1 \leq s \leq N$, there exist a metric g_s on $\mathcal{F}_s$ and some function γ_s such that (3.7) holds.*

Proof. Since by construction for any $s_1 \neq s_2$ the distributions D_{s_1} and D_{s_2} are orthogonal w.r.t. the metric G_1, the relation (3.7) is equivalent to the fact that for any s, $1 \leq s \leq N$, there exists the metric g_s on $\mathcal{F}_s$ and the function γ_s such that

$$\forall Y \in D_s(q) \quad G_{1q} = \gamma_s g_{s_{pr_s q}}\big((d(pr_s)_q Y\big) \tag{3.11}$$

If $\dim \mathcal{F}_s = 1$ (or, equivalently, $|I_s| = 1$), then relation (3.11) holds automatically for some g_s on $\mathcal{F}_s$ and some function γ_s, because all quadratic forms of one variable are proportional. Let us prove (3.11) in the general case. First, as g_s we can take the restriction $G_1\Big|_{\mathcal{F}_s(q_0)}$ of G_1 to $\mathcal{F}_s(q_0)$, i.e.,

$$g_s = G_1\Big|_{\mathcal{F}_s(q_0)} \tag{3.12}$$

Fix some point $q_1 \in \mathcal{F}_s(q_0)$ and denote by $\mathcal{G}_s(q_1)$ the integral manifold of the distribution $\mathrm{span}\{D_l : 1 \leq l \leq N, l \neq s\}$, passing through q_1. Fix some vector $v \in D_s(q_1)$ such that $g_s(v, v) = 1$. By construction for any $q \in \mathcal{G}_s(q_1)$ there exist a unique vector $Y_v(q) \in D_s(q)$ such that $d(pr_s)_q Y_v(q) = v$. Denote by ε_v the following function on $\mathcal{G}_s(q_1)$

$$\varepsilon_v(q) \stackrel{def}{=} G_{1q}\big(Y_v(q), Y_v(q)\big), \quad q \in \mathcal{G}_s(q_1) \tag{3.13}$$

It is clear that the relation (3.11) is equivalent to the fact that the function ε_v does not depend on the choice of the unit vector v from $D_s(q_1)$. Then in order to obtain (3.11) on $\mathcal{G}_s(q_1)$, we will put

$$\gamma_s = \varepsilon_v. \tag{3.14}$$

Let us prove that the function ε_v does not depend on unit vector v from $D_s(q_1)$. Fix some $l \neq s$ and some vector field $Z \in D_l$, which is unit w.r.t. the metric G_1, i.e. $G_1(Z,Z) = 1$. For some $j \in I_s$, $i \in I_l$ take an adapted frame $(X_1, \ldots, X_n)$ such that

$$X_j(q) = \varepsilon_v^{-1/2} Y_v(q), \quad \forall q \in \mathcal{G}_s(q_1), \tag{3.15}$$

$$X_i = Z. \tag{3.16}$$

First by construction one has

$$c_{ji}^j = -\frac{1}{2}\frac{X_i(\varepsilon_v)}{\varepsilon_v} \tag{3.17}$$

Indeed, let $(x_1, \ldots, x_n)$ be coordinates of Corollary 3 and suppose that $v = \sum_{k \in I_s} v_k \frac{\partial}{\partial x_k}$. Then by (3.15) on $\mathcal{G}_s(q_1)$ the fields X_j with $j \in I_s$ have the form

$$X_j = \varepsilon_v^{-1/2} \sum_{k \in I_s} v_k \frac{\partial}{\partial x_k}, \tag{3.18}$$

while by construction $X_i \in \text{span}\left(\frac{\partial}{\partial x_{\bar{l}}}\right)_{\bar{l} \notin I_s}$, which together with (3.18) implies (3.17). On the other hand, by (2.46) we have

$$c_{ji}^j = \frac{1}{2} X_i\left(\frac{\lambda_s}{\lambda_l}\right)\left(1 - \frac{\lambda_s}{\lambda_l}\right)^{-1}, \tag{3.19}$$

where as before $\lambda_s(q)$, $\lambda_l(q)$ are the eigenvalues of the transition operator S_q, corresponding to the eigenspaces $D_s(q)$ and $D_l(q)$. So, from (3.17), (3.19), (3.16), and definition of ϵ_v it follows that

$$\frac{Z(\varepsilon_v)}{\varepsilon_v} = -Z\left(\frac{\lambda_s}{\lambda_l}\right)\left(1 - \frac{\lambda_s}{\lambda_l}\right)^{-1}, \tag{3.20}$$

$$\varepsilon_v(q_1) = 1 \tag{3.21}$$

The right-hand side of (3.20) does not depend on the choice of the vector v. Hence from (3.20)-(3.21) it follows that on the curve $e^{tZ}q_1$ the function ϵ_v does not depend on the choice of the vector v. Note that any point of $\mathcal{G}_s(q_1)$ can be connected with q_1 by some finite concatenation of the integral curves of the fields $\pm Z$, where $Z \in D_l$, $l \neq s$. Therefore by induction on the number of "switches", one gets from (3.20) that on the manifold $\mathcal{G}_s(q_1)$ the function ε_v does not depend on the choice of the vector v. Defining γ_s, as in (3.14), we obtain (3.11) on $\mathcal{G}_s(q_1)$ and hence on U, which completes the proof of Lemma 7.□

Lemma 8 *There exist functions β_s on $\mathcal{F}_s(q_0)$ such that (3.9) holds.*

Proof. Let, as above, $(x_1, \ldots, x_n)$ be some coordinates from Corollary 3. Denote by χ_s the following $|I_s|$-tuple:

$$\chi_s = \{x_i\}_{i \in I_s}. \tag{3.22}$$

Since by construction

$$\text{span}\,\{X_i\}_{\in I_s} = \text{span}\,\{\frac{\partial}{\partial x_i}\}_{i \in I_s} = D_s, \tag{3.23}$$

relations (2.47) and (2.48) are equivalent to the following relations respectively

$$\forall 1 \le s \ne l \le N,\ i \in I_s: \quad \frac{\partial}{\partial x_i}\left(\frac{\lambda_l^2}{\lambda_s}\right) = 0, \tag{3.24}$$

$$\forall 1 \le s, l, r \le N,\ l \ne s,\ r \ne s,\ i \in I_s: \quad \frac{\partial}{\partial x_i}\left(\frac{\lambda_l}{\lambda_r}\right) = 0. \tag{3.25}$$

First suppose that $N = 2$. Then from (3.24) there exist functions $\bar\beta_s(\chi_s)$, $s = 1, 3$ such that

$$\frac{\lambda_2^2}{\lambda_1} = \bar\beta_2(\chi_2), \quad \frac{\lambda_1^2}{\lambda_2} = \bar\beta_1(\chi_1), \tag{3.26}$$

which easily implies (3.9), if we take $\beta_1 = \bar\beta_1^{1/3}$, $\beta_2 = \bar\beta_2^{1/3}$. For $N > 2$ a standard analysis of conditions (3.25) implies that there exist functions $\beta_s(\chi_s)$ such that

$$\frac{\lambda_s(q)}{\lambda_l(q)} = \frac{\beta_s(\chi_s)}{\beta_l(\chi_l)} \tag{3.27}$$

Substituting the last relation in (2.47) one can obtain easily that

$$\frac{\partial}{\partial x_j}\left(\frac{\lambda_s(q)}{\beta_l(\chi_l)}\right) = 0, \quad j \in I_l,\ l \ne s \tag{3.28}$$

Using standard arguments of "separation of variables" for the last equations, one can easily conclude that there exist a function $\sigma(\chi_s)$ such that

$$\lambda_s = \sigma(\chi_s) \prod_{l \ne s} \beta_l(\chi_l). \tag{3.29}$$

Substituting the last equation to (3.27) we obtain that

$$\frac{\sigma_s(\chi_s)}{\sigma_l(\chi_l)} = \frac{\beta_s^2(\chi_s)}{\beta_l^2(\chi_l)},$$

which in turn implies that $\sigma_i = C\beta_i^2$ for some constant $C > 0$. Replacing functions β_i by $k\beta_i$ for some constant $k > 0$ one can make $C = 1$. So,

$$\lambda_s = \beta_s(\chi_s) \prod_{l=1}^{N} \beta_l(\chi_l), \tag{3.30}$$

which is equivalent to (3.9). $\square$

Lemma 9 *If* $\dim \mathcal{F}_s > 1$, *then* λ_s *is constant on each leaf of the foliation* $\mathcal{F}_s$

Proof. Taking the coefficients of u_i^2, $i \ne j$, from (2.60) and using (3.1), we obtain the following relation

$$X_j(\alpha_i^2) = 2c_{ji}^i(\alpha_j^2 - \alpha_i^2) \quad i \ne j. \tag{3.31}$$

Note that identity (3.31) is stronger than identity (2.58): in the first identity we assume that the corresponding indices are different, while in the second one we assume that the corresponding eigenvalues are different. Take any pair of indices $i, j \in I_s$ such that $i \ne j$ (by assumption $|I_s| > 1$ it is possible). Applying (3.31) and using the fact that $\alpha_i = \alpha_j = \lambda_s^{1/2}$, we get $X_j(\lambda_s) = 0$ for any $j \in I_s$, which implies the statement of the lemma. $\square$

Remark 3 *The functions β_s from relation (3.9) have the intrinsic meaning, because they can be expressed by the eigenvalues of the transition operator S_q in the following way*

$$\beta_s \circ pr_s = \lambda_s^{\frac{N-1}{N+1}} \Big(\prod_{l \neq s} \lambda_l\Big)^{-\frac{2}{N+1}} \tag{3.32}$$

From the previous lemma and (3.9) it follows immediately the following

Corollary 4 *If $\dim \mathcal{F}_s > 1$, then the function β_s is constant.*

To complete the "only if" part it remains to prove relation (3.10). For this, combining (3.14), (3.17), and (3.19), then taking into account (3.23) and (3.27), one obtains without difficulties

$$\forall 1 \leq s \neq l \leq N, \ i \in I_l : \quad \frac{\partial}{\partial x_i} \ln \gamma_s = \frac{\partial}{\partial x_i} \ln \left| \frac{\beta_s(\chi_s)}{\beta_l(\chi_l)} - 1 \right|. \tag{3.33}$$

Again using standard "separation of variables" arguments we get from the last relations that there exist one-valuable functions $\omega_s(\chi_s)$ such that

$$\gamma_s = \omega_s(\chi_s) \prod_{l \neq s} \left| \frac{1}{\beta_l(\chi_l)} - \frac{1}{\beta_s(\chi_s)} \right|. \tag{3.34}$$

Finally note that by a change of coordinates of the type $\chi_s \mapsto F_s(\chi_s)$ we can make $\omega_s \equiv 1$ for any $1 \leq s \leq N$, which together with (3.34) implies (3.10). This completes the proof of the "only if" part.

Note that in the proof of the "only if" part we actually have used all information, which can be obtained from relations (3.1) (the only group of coefficients in (2.60) that we did not exploit are coefficients of $u_i u_j$ with $i \neq j$, but the identities that they produce from (3.1) are equivalent to identities (3.31), which was obtained by exploiting another group of coefficients). Therefore by Lemma 4 the conditions of the theorem are not only necessary, but also sufficient. The proof of the theorem is completed. □

For metrics on surfaces Levi-Civita's theorem is called also Dini's Theorem, because Dini obtained it first in [2].

4 The case of corank one distributions

In the present section we investigate the problem of geodesic equivalence of sub-Riemannian metrics on a distribution D of corank 1, especially, if D is contact or quasi-contact. From the beginning we work in the neighborhood of regular point q_0, extending then the results to the non-regular points by the limiting process, when it is possible.

Let the functions R_j and Q_{jk} be as in (2.18) and (2.19) respectively. All these functions are polynomials on the fibers. In general, these functions depend on the choice of the adapted frame to the pair of the metrics (G_1, G_2).

Definition 6 *We will say that the ordered pair (G_1, G_2) of sub-Riemannian metrics on the distribution D satisfies the second divisibility condition on an open set U, if there exist an adapted frame to the pair (G_1, G_2) in U such that for any $q \in U$ on the fiber T_q^*U the polynomial R_j is divided by the polynomial Q_{jm+1} for any index j such that $Q_{jm+1} \not\equiv 0$ on T_q^*U, $1 \leq j \leq m$.*

Note that $\bar{c}_{ji}^{m+1} = \frac{1}{\alpha_i \alpha_j} c_{ji}^{m+1}$ for any i, j such that $1 \le i \le j$. Therefore

$$Q_{jm+1} = \frac{1}{\alpha_j} \sum_{i=1}^{m} c_{ji}^{m+1} u_i. \tag{4.1}$$

Proposition 8 *Suppose that for given two sub-Riemannian metrics G_1 and G_2 on corank 1 distribution D and for some open set U of regular point q_0 there exists an orbital diffeomorphism of the extremal flows of these metrics in some open set $\mathcal{B}$ in $H_1 \cap T^*U$, $\pi(\mathcal{B}) = U$. Then the pair (G_1, G_2) satisfies the second divisibility condition on U.*

Proof. Fix some index j, $1 \le j \le m$, such that

$$Q_{jm+1} \not\equiv 0. \tag{4.2}$$

Substituting (2.37) into (2.21) we obtain

$$-\frac{\vec{h}_1(Q_{jm+1}) R_j}{\alpha_j Q_{jm+1}^2 \mathcal{P}^{1/2}} = \frac{\text{polynomial}}{Q_{jm+1} \mathcal{P}^{3/2}}$$

or, equivalently,

$$\frac{\mathcal{P} \vec{h}_1(Q_{jm+1}) R_j}{Q_{jm+1}} = \text{polynomial}. \tag{4.3}$$

Positive definite quadratic form $\mathcal{P}$ cannot be divided by Q_{jm+1}, which is linear function with real coefficients. Let us prove that Q_{jm+1} does not divide $\vec{h}_1(Q_{jm+1})$. Assuming the converse, one can conclude that the coefficients of $u_j u_{m+1}$ in the quadratic polynomial $\vec{h}_1(Q_{jm+1})$ has to be equal to zero (because Q_{jm+1} does not depend both on u_j and on u_{m+1}). On the other hand, from (2.24) and (4.1) it is not hard to get that this coefficient is equal to

$$-\frac{1}{\alpha_j} \sum_{i=1}^{m} \left(c_{ji}^{m+1}\right)^2.$$

Hence $c_{ji}^{m+1} = 0$ for all $1 \le i \le m$, which contradicts the assumption (4.2). So, relation (4.3) yields that R_j has to be divided by Q_{jm+1}, i.e., the second divisibility condition holds.□

Proposition 9 *Suppose that for given two sub-Riemannian metrics G_1 and G_2 on some $(m, m+1)$-distribution D and for some open set U there exists an orbital diffeomorphism of the extremal flows of these metrics in some open set $\mathcal{B}$ in $H_1 \cap T^*U$, $\pi(\mathcal{B}) = U$. Suppose also that there exists the basis $(X_1, \ldots, X_m)$ of D adapted to the ordered pair (G_1, G_2), and the transition operator S_q has the form $S_q = \operatorname{diag}(\alpha_1^2(q), \ldots, \alpha_m^2(q))$ in this basis ($\alpha_i > 0$). Then the following two statements hold*

1. *If*
$$I \stackrel{def}{=} \Big\{ j \in \{1, \ldots, m\} : [X_j, D](q) \not\subset D(q) \ \forall q \in U \Big\}, \tag{4.4}$$
then $\alpha_i = \alpha_j$ in U for all $i, j \in I$;

2. *If $\alpha \stackrel{def}{=} \alpha_j$, $j \in I$, and $\bar{I} = \Big\{ j \in \{1, \ldots, m\} : \alpha_j = \alpha \Big\}$, then*
$$\forall j \in \bar{I} : \qquad X_j(\alpha) = 0 \tag{4.5}$$

Proof. By Proposition 8 for any $j \in I$ the polynomial R_j is divided by $\alpha_j Q_{jm+1}$. But by (2.37) the polynomial $\frac{R_j}{\alpha_j Q_{jm+1}}$ does not depend on $j \in I$ (because it is equal to $\sqrt{\mathcal{P}}\Phi_{m+1}$). In other word,

$$R_j = \Big(\sum_{i=1}^{m+1} r_i u_i\Big)\alpha_j Q_{jm+1}, \tag{4.6}$$

where coefficients r_i do not depend on $j \in I$. As a consequence of the last identity and (2.37) one has

$$\Phi_{m+1} = \frac{\sum_{i=1}^{m+1} r_i u_i}{\sqrt{\mathcal{P}}}. \tag{4.7}$$

Using (2.60) and (4.1), one can compare the coefficients of $u_i u_{m+1}$, $1 \le i \le m$ in both sides of (4.6) to get

$$\alpha_j^2 c_{ji}^{m+1} = r_{m+1} c_{ji}^{m+1}.$$

Since by definition for any $j \in I$ there exist $1 \le i \le m$ such that $c_{ji}^{m+1} \neq 0$, then

$$\forall j \in I: \quad \alpha_j^2 = r_{m+1} \tag{4.8}$$

In other words, α_j does not depend on $j \in I$, which concludes the proof of the first statement of the proposition.

Let us prove the second statement. From (2.60) and the fact that $\alpha_j = \alpha_i = \alpha$ for all $i \in I$ it follows that

$$\forall i \in I: \Big(\text{the coefficient of } u_i^2 \text{ in } R_j\Big) = -\frac{1}{2}X_j(\alpha^2). \tag{4.9}$$

If $j \in \bar{I}\backslash I$, then $Q_{jm+1} = 0$ and by identity (2.20) we have $R_j = 0$, which together with (4.9) implies that $X_j(\alpha^2) = 0$.

If $j \in I$, then comparing the coefficients of u_i^2, $i \in I$, $i \neq j$ in both sides of (4.6) and using relations (4.9), (4.1), we obtain

$$\frac{1}{2}X_j(\alpha^2) = r_i c_{ij}^{m+1}. \tag{4.10}$$

Substituting identity (4.7) into identity (2.21) with $s = m+1$, then using (2.53), and finally multiplying both sides on $\sqrt{\mathcal{P}}$, we get

$$\vec{h}_1\Big(\sum_{i=1}^{m+1} r_i u_i\Big) - \frac{1}{2}\Big(\sum_{j=1}^{m}\frac{X_j(\alpha_j^2)}{\alpha_j^2}u_j\Big)\sum_{i=1}^{m+1} r_i u_i - Q_{m+1\,m+1}\sum_{i=1}^{m+1} r_i u_i = \sum_{k=1}^{m} Q_{m+1\,k}\alpha_k u_k \tag{4.11}$$

Comparing the coefficients of $u_j u_{m+1}$, $j \in I$ in both sides of (4.11) one can obtain without difficulties that

$$\sum_{i=1}^{m} r_i c_{ij}^{m+1} + \frac{1}{2}X_j(\alpha^2) = 0,$$

which together with (4.10) implies that $\frac{n_j+1}{2}X_j(\alpha^2) = 0$, where n_j is the number of indices i, $1 \le i \le m$ such that $c_{ij}^{m+1} \neq 0$. Therefore $X_j(\alpha^2) = 0$ for all $j \in I$. The proof of the second statement is also completed. □

As a direct consequence of Proposition 3 and the previous proposition we obtain the following

Theorem 2 *If two sub-Riemannian metrics G_1 and G_2, defined on a contact distribution D, are geodesically equivalent at some point q_0, then they are constantly proportional in some neighborhood of q_0.*

Proof. First note that it is sufficient to prove this theorem for regular q_0: using the density of the set of regular points (Proposition 1), one can extend the theorem to the non-regular points by passing to the limit. If q_0 is regular, then there exists the basis $(X_1, \dots, X_m)$ of D adapted to the ordered pair (G_1, G_2). Let, as before, the transition operator S_q has the form $S_q = \text{diag}\,(\alpha_1^2(q), \dots, \alpha_m^2(q))$ in this basis ($\alpha_i > 0$). In the case of the contact distribution the set I, defined by (4.4), coincides with $\{1, \dots, m\}$. Therefore, by consecutive use of Propositions 3 and 9 we obtain that there exists the function α such that $\alpha_i = \alpha$ and $X_i(\alpha) = 0$ for any i, $1 \le i \le m$. This together with the fact that contact distribution is bracket generating implies that $\alpha_i = \alpha = \mathit{const}$ for any i, $1 \le i \le m$, which concludes the proof of the theorem. $\Box$

For $(2,3)$-distributions we can extend the last result from contact to all bracket-generating distributions, because the set of points, where bracket-generating $(2,3)$- distributions are contact, is open and dense. Namely, we have the following

Corollary 5 *If two sub-Riemannian metrics G_1 and G_2, defined on a bracket-generating $(2,3)$-distribution D, are geodesically equivalent at some point q_0, then they are constantly proportional in some neighborhood of q_0.*

Now consider the case of the quasi-contact distribution D. The following theorem gives the classification of all geodesically equivalent sub-Riemannian metrics, defined on such distribution:

Theorem 3 *Suppose that G_1 and G_2 are two sub-Riemannian metrics on the quasi-contact distribution D such that $G_2 \not\equiv \text{const}\, G_1$. Assume also that the vector field X is tangent to the abnormal line distribution of D and unit w.r.t. the metric G_1 (i.e., ${G_1}_q(X, X) = 1$). Then the metrics G_1 and G_2 are geodesically equivalent at the point q_0 if and only if in some neighborhood U of q_0 the following four conditions hold simultaneously:*

1. *If*
$$D_i(q) = \{v \in D(q) : {G_i}_q(v, X) = 0\}, \quad i = 1, 2, \tag{4.12}$$
then $D_1(q) = D_2(q)$ and the distribution D_1^2 is codimension 1 integrable distribution (here $D_1^2 = D_1 + [D_1, D_1]$);

2. *If $\mathcal{F}$ is the foliation of the integral hypersurfaces of the distribution D_1^2, then the flow e^{tX} generated by the vector field X preserves the foliation $\mathcal{F}$, i.e., it maps any leaf of $\mathcal{F}$ to a leaf of $\mathcal{F}$;*

3. *There exists the one-variable function $\beta(t)$, $\beta(0) = 1$, such that if $\mathcal{F}_0$ is the leaf of the foliation $\mathcal{F}$ passing through q_0 and $G_1\Big|_{e^{tx}\mathcal{F}_0}$ is the restriction of the metric G_1 to the leaf $e^{tX}\mathcal{F}_0$, then*
$$G_1\Big|_{e^{tX}\mathcal{F}_0} = \beta(t)\Big((e^{-tX})^* G_1\Big)\Big|_{e^{tX}\mathcal{F}_0}; \tag{4.13}$$

4. *There exist two constants $C_1 > 0$ and $C_2 > -1$, $C_2 \neq 0$, such that if $\mathcal{F}_0$ is as before and $G_2\Big|_{e^{tx}\mathcal{F}_0}$ is the restriction of the metric G_2 to the leaf $e^{tX}\mathcal{F}_0$, then*
$$G_2\Big|_{e^{tX}\mathcal{F}_0} = \frac{C_1}{1 + C_2\beta(t)} G_1\Big|_{e^{tX}\mathcal{F}_0}, \tag{4.14}$$
$$\forall q \in e^{tX}\mathcal{F}_0 : \quad {G_2}_q\big(X(q), X(q)\big) = \frac{C_1}{\big(1 + C_2\beta(t)\big)^2}. \tag{4.15}$$

Before proving Theorem 3, let us make some remarks. According to this theorem for the quasi-contact distribution D the pair (G_1, G_2) of constantly non-proportional geodesically equivalent metrics at the point q_0 is uniquely determined by fixing

a) a vector field X tangent to the abnormal line distribution of D;

b) a hypersurface $\mathcal{F}_0$, passing through q_0 and transversal to the abnormal line distribution of D;

c) a sub-Riemannian metric $\bar{G}$ on the contact distribution $\bar{D}$, defined on the hypersurface $\mathcal{F}_0$ as follows: $\bar{D}(q) = D(q) \cap T_q\mathcal{F}_0$, $q \in \mathcal{F}_0$;

d) a one-variable function $\beta(t)$ with $\beta(0) = 1$;

e) two constants C_1, C_2, where $C_1 > 0$ $C_2 > -1$, and $C_2 \neq 0$.

The metrics G_1 can be uniquely recovered from the data of a)-d). For this we extend the distribution $\bar{D}$ and the metric $\bar{G}$ on $\bar{D}$ from $\mathcal{F}_0$ to M by the flow e^{tX}. Namely, we set

$$\forall q \in \mathcal{F}_0: \quad \bar{D}(e^{tX}q) = (e^{tX})_*\bar{D}(q), \quad \bar{G}_{e^{tX}q}(v, w) = \bar{G}_q\big((e^{-tX})_*v, (e^{-tX})_*w\big), \;\; v, w \in \bar{D}(e^{tX}q).$$

Then the metric G_1 is uniquely defined by the following two conditions:

- for any $q \in e^{tX}\mathcal{F}_0$ on the subspace $\bar{D}(q)$ the metric G_1 coincides with $\bar{G}$ multiplied by the factor $\beta(t)$
- for any q the vector $X(q)$ is unit and orthogonal to $\bar{D}(q)$ w.r.t. G_1.

In particular, it shows that the metrics on quasi-contact distributions, admitting constantly non-proportional geodesically equivalent metrics, are very special. The metric G_2 is uniquely determined by G_1 and two constants C_1 and C_2 with the properties prescribed in e). In other words, if the metric G_1 admits constantly non-proportional geodesically equivalent metrics, then the set of such metrics is two-parametric. Note also that if one takes $C_2 = 0$ in statement 4 of Theorem 3, then the metrics are constantly proportional.

Proof of Theorem 3. Let us prove the "only if" part. Let the metrics G_1 and G_2 be geodesically equivalent. First suppose that q_0 is regular point w.r.t. the pair (G_1, G_2). As before let $(X_1, \ldots, X_m)$ be a basis of D adapted to the ordered pair (G_1, G_2) and suppose that the transition operator $S_q = \mathrm{diag}\big(\alpha_1^2(q), \ldots, \alpha_m^2(q)\big)$ (where $\alpha_i > 0$) w.r.t. the basis $(X_1, \ldots, X_m)$.

First note that the field X has to coincide with one of the fields X_i, $1 \leq i \leq m$. Otherwise, the set I, defined by (4.4), coincides with $\{1, \ldots, m\}$. Then by the same arguments, as in the proof of Theorem 2, we obtain that the metrics G_1 and G_2 are constantly proportional, which contradicts our assumptions. Without loss of generality, it can be assumed that $X = X_m$. Secondly by Proposition 9 for any $1 \leq i, j \leq m-1$ we have $\alpha_i = \alpha_j$. In the sequel we set $\alpha_i = \alpha$ for $1 \leq i \leq m-1$.

Since the field $X_m = X$ has no singularities, by passing to the limit one obtains that the adapted basis with the same properties exists also in a neighborhood of non-regular points w.r.t. to the pair (G_1, G_2). Moreover, $\alpha_m \neq \alpha$. Indeed, assuming the converse we obtain from the statement 2 of Proposition 9 that the set $\bar{I} = \big\{j \in \{1, \ldots, m\} : \alpha_j = \alpha\big\}$ coincides with $\{1, \ldots, m\}$. But from this again by the same arguments, as in the proof of Theorem 2, we obtain that the metrics G_1 and G_2 are constantly proportional, which contradicts our assumptions. Actually, we have shown that for geodesically equivalent metrics q_0 is always regular: in some neighborhood of q_0 the number of distinct eigenvalues of the transition operator is constant and equal either to 1 (in this case the metrics are constantly proportional) or to 2. Besides, if D_1 and D_2 are as in (4.12), then

$$D_1 = D_2 = \mathrm{span}(X_1, \dots, X_{m-1}).$$

From (2.47) it follows that $X_i\Big(\frac{\alpha_m^2}{\alpha}\Big) = 0$ for all $1 \le i \le m-1$, which together with (4.5) implies

$$\forall 1 \le i \le m-1: \quad X_i(\alpha_m) = 0. \tag{4.16}$$

Replacing the (4.5) and (4.16) in (2.46), we obtain also that

$$\forall 1 \le i \le m-1: \quad c_{mi}^m = 0. \tag{4.17}$$

Let us complete the adapted basis $(X_1, \dots, X_m)$ somehow to the frame $(X_1, \dots, X_{m+1})$.

Lemma 10 *The distribution $D_1^2 = D_1 + [D_1, D_1]$ is integrable.*

Proof. Using (2.60) and (4.1), let us compare the coefficients of $u_i u_m$, $1 \le i \le m-1$ in both sides of (4.6), where $1 \le j \le m-1$. As a result, we get easily

$$\forall 1 \le i \ne j \le m-1: \quad (\alpha^2 - \alpha_m^2) c_{ji}^m = r_m c_{ji}^{m+1} + r_i c_{jm}^{m+1}.$$

But by construction $m \notin I$, i.e., $c_{jm}^{m+1} = 0$ for all $1 \le j \le m-1$. Therefore the last relation is equivalent to the following identity:

$$\forall 1 \le i \ne j \le m-1: \quad c_{ji}^m = \frac{r_m}{\alpha^2 - \alpha_m^2} c_{ji}^{m+1}. \tag{4.18}$$

Hence $[X_i, X_j] \in \mathrm{span}\Big(X_1, \dots, X_{m-1}, \frac{r_m}{\alpha^2 - \alpha_m^2} X_m + X_{m+1}\Big)$ for all $1 \le i, j \le m-1$ or , equivalently,

$$D_1^2 = \mathrm{span}\Big(D_1, \frac{r_m}{\alpha^2 - \alpha_m^2} X_m + X_{m+1}\Big). \tag{4.19}$$

To prove the lemma it is sufficient to prove that

$$\forall 1 \le i \le m-1: \quad \Big[X_i, \frac{r_m}{\alpha^2 - \alpha_m^2} X_m + X_{m+1}\Big] \in \mathrm{span}\Big(D_1, \frac{r_m}{\alpha^2 - \alpha_m^2} X_m + X_{m+1}\Big). \tag{4.20}$$

Using (4.5) and (4.16), it is easy to show that (4.20) is equivalent to the following identity

$$X_i(r_m) + r_m c_{mi}^m + c_{m+1i}^m(\alpha^2 - \alpha_m^2) - r_m c_{m+1\,i}^{m+1} = 0 \tag{4.21}$$

Let us prove identity (4.21). First note that from (4.5) and (4.10) it follows easily that $r_i = 0$ for $1 \le i \le m-1$ (here we use also the fact that for given i, $1 \le i \le m-1$, there exist j, $1 \le j \le m-1$, such that $c_{ij}^{m+1} \ne 0$). From this and (4.5) it follows that the identity (4.11) can be rewritten in the following form:

$$\vec{h}_1\Big(\sum_{i=m}^{m+1} r_i u_i\Big) - \frac{1}{2}\frac{X_m(\alpha_m^2)}{\alpha_m^2} u_m \sum_{i=m}^{m+1} r_i u_i - Q_{m+1\,m+1} \sum_{i=m}^{m+1} r_i u_i = \sum_{k=1}^{m} Q_{m+1\,k} \alpha_k u_k \tag{4.22}$$

Comparing the coefficients of $u_i u_m$, $1 \le i \le m-1$ in both sides of (4.22) by use of (2.24) and (2.19) it is not difficult to obtain

$$X_i(r_m) + r_m c_{mi}^m + r_{m+1}(c_{m+1i}^m + c_{m+1m}^i) - r_m \bar{c}_{m+1\,i}^{m+1} \alpha = (\bar{c}_{m+1m}^i + \bar{c}_{m+1i}^m)\alpha\alpha_m \tag{4.23}$$

From (2.61) and (2.22) it follows that $\bar{c}_{m+1\,i}^{m+1} = \frac{1}{\alpha} c_{m+1\,i}^{m+1}$, $\bar{c}_{m+1m}^i = \frac{\alpha}{\alpha_m} c_{m+1m}^i$, and $\bar{c}_{m+1i}^m = \frac{\alpha_m}{\alpha} c_{m+1i}^m$, while by (4.8) we have $r_{m+1} = \alpha^2$. Substituting all this to (4.23) we get (4.21), which completes the proof of the lemma. □

Lemma 11 *If $\mathcal{F}$ is the foliation of the integral hypersurfaces of the distribution D_1^2, then the flow e^{tX} generated by the vector field X preserves the foliation $\mathcal{F}$.*

Proof. From the previous lemma it follows that in some neighborhood U of q_0 there exist coordinates $(x_1, \dots, x_{m+1})$ such that the leaves of $\mathcal{F}$ are $\{x_m = const\}$ and $X_m = \nu \frac{\partial}{\partial x_m}$ for some function ν. By construction, all vector fields X_i with $1 \le i \le m-1$ belong to $\mathrm{span}\big(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{m-1}}, \frac{\partial}{\partial x_{m+1}}\big)$. Therefore $c_{mi}^m = \frac{X_i(\nu)}{\nu}$ for all $1 \le i \le m-1$, which together with (4.17) implies that $X_i(\nu) = 0$ for all $1 \le i \le m-1$. Then ν is constant on each leaf of $\mathcal{F}$, which equivalent to the statement of the lemma. □

Lemma 12 *Relation (4.13) holds for some one-variable function $\beta(t)$.*

Proof. Actually the proof of this lemma is very similar to the proof of Lemma 7. Since the vector field $X = X_m$ satisfies $[X, D] \subset D$, the flow e^{tX} preserves the distribution D. This and the previous lemma implies that e^{tX} preserves also the distribution D_1 (note that by the previous lemma $D_1(q) = D(q) \cap T_q\mathcal{F}(q)$, where $\mathcal{F}(q)$ is the leaf of the foliation $\mathcal{F}$, passing through the point q).

Fix some point $q_1 \in \mathcal{F}_0$. Denote by L_{q_1} the abnormal extremal trajectory passing through q_1. Fix some vector $v \in D_s(q_1)$ such that $G_{1q_1}(v, v) = 1$. By construction for any point $q \in L_{q_1}$ such that $q = e^{tX}q_1$ there exist a unique vector $Y_v(q) \in D_1(q)$ such that $d(e^{-tX})_q Y_v(q) = v$. Denote by ε_v the following function on the curve L_{q_1}.

$$\varepsilon_v(q) \stackrel{def}{=} G_{1q}\big(Y_v(q), Y_v(q)\big), \quad q \in L_{q_1} \tag{4.24}$$

By the same arguments as in the proof of Lemma 7, we obtain that the function ε_v does not depend on the choice of the unit vector v from $D_1(q_1)$. It implies that for any q in some neighborhood U of q_0 (here any coordinate neighborhood from the proof of the previous lemma can be taken as U) there is $\beta(q)$ such that if $q = e^{tX}q_1$, where $q_1 \in \mathcal{F}_0$, then

$$G_1\Big|_{e^{tX}\mathcal{F}_0} = \beta\Big((e^{-tX})^* G_1\Big)\Big|_{e^{tX}\mathcal{F}_0} \tag{4.25}$$

Besides, similarly to (3.20)-(3.21), we have

$$\frac{X(\beta)}{\beta} = -X\left(\frac{\alpha^2}{\alpha_m^2}\right)\left(1 - \frac{\alpha^2}{\alpha_m^2}\right)^{-1}, \tag{4.26}$$

$$\beta\Big|_{\mathcal{F}_0} = 1. \tag{4.27}$$

Finally by (4.5) and (4.16) the functions α and α_m are constant on each leaf of the foliation $\mathcal{F}$. Therefore (4.26)-(4.27) implies that the function β is constant on each leaf of the foliation $\mathcal{F}$ too. This fact together with (4.25) implies (4.13).□

In order to complete the proof of the "only if" part it remains to prove identities (4.14) and (4.15). By (2.12) and statement 1 of Proposition 9

$$\forall q \in e^{tX}\mathcal{F}_0 : \quad G_{2q} = \alpha^2(q) G_{1q} \tag{4.28}$$

$$\forall q \in e^{tX}\mathcal{F}_0 : \quad G_{2q}\big(X(q), X(q)\big) = \alpha_m^2(q). \tag{4.29}$$

So, it remains to find the functions α and α_m. As was mentioned in the proof of the previous lemma, the functions α and α_m are constant on each leaf of the foliation $\mathcal{F}$. Besides, by (2.47) we have $X_m(\frac{\alpha^2}{\alpha_m}) = 0$. So,

$$\frac{\alpha^2}{\alpha_m} \equiv C, \tag{4.30}$$

where C is constant. Then from (2.46) it follows that for any j, $1 \le j \le m-1$

$$X_m\left(\frac{C}{\alpha_m}\right) = 2c^j_{jm}\left(1 - \frac{C}{\alpha_m}\right) \tag{4.31}$$

By Lemmas 10-12 we can choose the coordinates $(y_1, \dots, y_m, t)$ in a neighborhood of q_0 such that $q_0 = (0, \dots, 0)$ and

$$X_m = \frac{\partial}{\partial t}; \tag{4.32}$$

$$X_j = \beta(t)^{-1/2} \sum_{k=1}^{m} \nu_{jk} \frac{\partial}{\partial y_k}, \quad 1 \le j \le m-1. \tag{4.33}$$

As in (3.17) this yields that

$$c^j_{jm} = -\frac{1}{2}\frac{d}{dt}\ln\beta(t).$$

Substituting the last formula in (4.31), one can obtain without difficulties that

$$\alpha_m = \frac{C}{1 + C_2\beta(t)} \tag{4.34}$$

for some constant C_2, $C_2 > -1$, $C_2 \ne 0$. Then by (4.30)

$$\alpha^2 = C\alpha_m = \frac{C^2}{1 + C_2\beta(t)} \tag{4.35}$$

Setting $C_1 = C^2$ and substituting (4.35) and (4.34) into (4.28) and (4.29), we get (4.14) and (4.15). The proof of the "only if" part of the theorem is completed.

Note that in the proof of the "only if " part we actually have used all information, contained in (4.6), which is equivalent to (2.20). Also, it can be shown by direct check that if all conditions 1-4 of Theorem 3 hold then the identity (4.22) holds too (but this identity is equivalent to (2.21)). From this, Lemma 2, and Proposition 2 it follows that conditions 1-4 of the theorem are also sufficient for the geodesic equivalence of our metrics at q_0. $\square$

5 The case of Riemannian metrics on a surface near non-regular isolated point

In the present section for the Riemannian metrics on a surface we obtain the classification of geodesically equivalent pairs at non-regular point (the point of bifurcation of the eigenvalues of the transition operator). Namely, we consider the case when two Riemannian metrics on a surface are proportional in an isolated point. Since the set of all 2×2 symmetric matrices with the equal eigenvalues has codimension 2 in the set of all 2×2 symmetric matrices, we have that for generic pair of Riemannian metrics on a surface the set of points of their proportionality consists of isolated points. Therefore its is natural to consider the case when two Riemannian metrics on a surface are proportional in an isolated point. It turns out that Dini's Theorem (i.e., Levi-Civita's theorem in the case of a surface) can be naturally extended to this case.

First let us formulate Dini's Theorem in the case of non-proportional metrics and analyze its additional features.

Theorem 4 *(Dini's Theorem) Suppose that two Riemannian metrics G_1 and G_2 on a surface are non-proportional at some point q_0. Then they are geodesically equivalent at q_0 if and only if in some neighborhood of q_0, there exist coordinates (x_1, x_2), $q_0 = (x_1^0, x_2^0)$, and one-variable functions $\beta_1(x_1)$ and $\beta_2(x_2)$ ($\beta_1(x_1^0) < \beta_2(x_2^0)$) such that in this coordinates*

$$||\cdot||_1^2 = \left(\frac{1}{\beta_1(x_1)} - \frac{1}{\beta_2(x_2)}\right)(dx_1^2 + dx_2^2), \tag{5.1}$$

$$||\cdot||_2^2 = \beta_1(x_1)\beta_2(x_2)\left(\frac{1}{\beta_1(x_1)} - \frac{1}{\beta_2(x_2)}\right)\left(\beta_1(x_1)dx_1^2 + \beta_2(x_2)dx_2^2\right), \tag{5.2}$$

where $||v||_i^2 = G_i(v, v)$, $i = 1, 2$.

The coordinates, appearing in Theorem 4, will be called *Dini's coordinates* of the ordered pair of Riemannian metrics (G_1, G_2). The following lemma will be useful in the sequel

Lemma 13 *If (x_1, x_2) and $(\bar{x}_1, \bar{x}_2)$ are two Dini's coordinates of the ordered pair of Riemannian metrics (G_1, G_2) on the same neighborhood U, then $\bar{x}_i = \pm x_i + c_i$ some constants c_i, $i = 1, 2$.*

Proof. From Corollary 3 and the fact that in Theorem 1 we assume that $\beta_1(x_1^0) < \beta_2(x_2^0)$ it follows that the coordinate net of all Dini's coordinates on U coincide: $D_1 = \{dx_2 = 0\} = \{d\bar{x}_2 = 0\}$ is the line distribution of the eigenvectors of the transition operator, corresponding to its smallest eigenvalue, while $D_2 = \{dx_1 = 0\} = \{d\bar{x}_1 = 0\}$ is the line distribution of the eigenvectors of the transition operator, corresponding to its biggest eigenvalue. Hence the transition function between the coordinates has a form $x_i = \psi_i(\bar{x}_i)$, $i = 1, \dots n$. Then the first metric is written in the coordinates $(\bar{x}_1, \dots, \bar{x}_n)$ as follows:

$$||\cdot||_1^2 = \left(\frac{1}{\beta_1(\psi(\bar{x}_1))} - \frac{1}{\beta_2^2(\psi(\bar{x}_2))}\right)\sum_{j=1}^{2}(\psi_i'(\bar{x}_j))^2(d\,\bar{x}_j)^2.$$

By Remark 3 the coefficients of dx_j^2 in (5.1) do not depend on the choice of the Dini coordinates. Therefore $(\psi_i'(\bar{x}_j))^2 \equiv 1$, which implies the statement of the Lemma. $\square$

Recall that a Riemannian metric on a surface defines the canonical conformal structure: In a neighborhood of any point there is a coordinate system in which the Riemannian metric has the form

$$||\cdot||^2 = \mu(x, y)(d\,x^2 + d\,y^2). \tag{5.3}$$

Such coordinates are called *isothermal* (see, for example, [8] or [3]). The transition function from one isothermal coordinates to some other is conformal mapping, up to the orientation, so the set of all charts with isothermal coordinates defines the conformal structure. Note that by (5.1) all Dini's coordinates are isothermal w.r.t. the first metric G_1.

Now suppose that the Riemannian metrics G_1 and G_2 are proportional at some isolated point q_0 and geodesic equivalent in a neighborhood of this point. Choose in a neighborhood B of q_0 some isothermal coordinates (x, y) w.r.t. the first metric G_1. Also, we can assume that the metrics are geodesic equivalent in B (otherwise we can take a smaller neighborhood). By above for any $q \in B$ in a neighborhood B_q of q there exist Dini's coordinates (u, v) of the ordered pair (G_1, G_2). We also can take them such that they define the same orientation as (x, y). The pair $(B_q, u(x, y) + iv(x, y)$ is a function element of an analytic function. Taking one of such function elements and using the standard procedure of the analytic continuation, we get the analytic function F in the punctured neighborhood $B\backslash q_0$ such that each of its function elements defines the transition function from the chosen isothermal coordinates (x, y) to Dini's coordinates of the ordered pair (G_1, G_2) in the neighborhood of this function element. The function F will be called a *Dini transition function* of the ordered pair of geodesic equivalent Riemannian metrics from the given isothermal coordinates (x, y). The following theorem gives the characterization of Dini's transition functions at an isolated point of the proportionality of the metrics:

Theorem 5 *If $F(z)$ is some Dini transition function of the ordered pair of Riemannian metrics, which are proportional at an isolated point q_0 and geodesic equivalent in a neighborhood of this point, then the function $(F')^2$ has a pole of order 1 or 2 at q_0. Besides, if $(F')^2$ has a pole of order 2 at q_0, then the principle negative coefficient in its Laurent expansion at q_0 has to be real.*

Proof. First note that the function $(F')^2$ is an one-valued function on some punctured neighborhood $B\backslash q_0$ of q_0. Indeed, by Lemma 13 the function elements (V, F_1) and (V, F_2) of F (with the common neighborhood of definition) satisfy

$$F_1(z) \equiv \pm F_2(z) + c, \quad z = x + iy, \tag{5.4}$$

where c is some complex constant. This implies that $(F_1')^2 \equiv (F_2')^2$.

Now let us prove that $(F')^2$ has a pole at q_0. Indeed, suppose that in the original coordinates the metric G_1 satisfies (5.3) with some function μ. Writing the metric G_1 in Dini's coordinates, we obtain that

$$\mu = \Big(\frac{1}{\beta_1} - \frac{1}{\beta_2}\Big)|F'|^2, \tag{5.5}$$

where the functions β_i are as in Theorem 4. The functions β_i are expressed by the eigenvalues λ_j of the transition operator as in (3.32) with $n = 2$. The condition of the proportionality of the metrics at q_0 implies that

$$\beta_1(q_0) = \beta_2(q_0) \tag{5.6}$$

(because $\lambda_1(q_0) = \lambda_2(q_0)$). From this, (5.5) and the fact that the function μ has no singularity at q_0 it follows that $\lim_{z \to q_0} |F'(z)|^2 = \infty$, i.e. $(F')^2$ has a pole at q_0.

Although the function F is in general multiple-valued, by (5.4) the families of the level sets of the function $\operatorname{Re} F$ (the function $\operatorname{Im} F$) for all its branches coincide. By construction the function β_1 is constant on the level set of $\operatorname{Re} F$, while the function β_2 is constant on the level set of $\operatorname{Im} F$. Using this fact it is not difficult to prove that the order of pole of $(F')^2$ at q_0 is not greater than 2. Assuming the converse, we obtain that the function F^2 also has a pole at q_0. So, F^2 maps a puncture neighborhood of q_0 onto the neighborhood of infinity and also sends the point q_0 to ∞. But any level set of $\operatorname{Re} F$ is the preimage w.r.t. the mapping F^2 of some parabola of the type $u = c^2 - \frac{v^2}{4c^2}$ on the plane w, where $w = F^2(z)$, $w = u + iv$. Such parabolas have ∞ as an accumulation point. Hence q_0 is the accumulation point of any level set of the function $\operatorname{Re} F$. This together with the fact that β_1 is constant on the level set of $\operatorname{Re} F$ and continuous at q_0 implies that β_1 is identically equal to some constant C_1 in a neighborhood of q_0. In the same way, β_2 is identically equal to some constant C_2 there. Moreover, by (5.6) $C_1 = C_2$. But it means that our metrics are proportional in the neighborhood of q_0, which contradicts our assumptions. So, the order of pole of $(F')^2$ at q_0 is not greater than 2.

To complete the proof of the theorem it remains to show that if $(F')^2$ has a pole of order 2 at q_0, then the principle negative coefficient in its Laurent expansion at q_0 is real. Indeed, if $(F')^2$ has a pole of order 2 with the principle negative coefficient a in its Laurent expansion at q_0, then F has the logarithmic singularity at q_0 with coefficient $\sqrt{a}$ near the logarithm. In this case after the appropriate change of independent variable z in a neighborhood of q_0 (i.e., conformal change of coordinates in a neighborhood of q_0) one can get $F(z) = \sqrt{a}\log z$, $q_0 = 0$. But if a is not real, then $\sqrt{a}$ is neither real nor pure imaginary. In this case all level sets of both ReF and ImF are spirals having q_0, as an accumulation point. As above, it implies that functions β_1 and β_2 are equal to the same constant, which is impossible. The proof of the theorem is completed. $\square$

According to the previous theorem only the following two situation are possible at an isolated point of proportionality of two metrics:

1) $(F')^2$ has a simple pole at $q_0 \Leftrightarrow F(z) = \sqrt{G(z)}$ for some analytic function $G(z)$, having at q_0 zero of order 1 (i.e., F has the "square root" singularity at q_0). In this case after the appropriate change of independent variable z in a neighborhood of q_0 one can get $F(z) = \sqrt{z}$, $q_0 = 0$;

2) F' has a simple pole at q_0 with real or pure imaginary residue at $q_0 \Leftrightarrow F(z)$ has the logarithmic singularity at q_0 with real or pure imaginary coefficient b near logarithm. In this case after the appropriate change of independent variable z in a neighborhood of q_0 one can get $F(z) = b\log z$, $q_0 = 0$, where b is real or pure imaginary constant. If b is real then the level sets of $\operatorname{Im} F(z)$ are the rays, starting at 0. Hence by the same argument as in the proof of the previous theorem we can conclude that the function β_2 is constant. Besides, the function β_1 in this case depends only on $|z|$ (here β_i as in Theorem 1). In the same way, if b is pure imaginary, then β_1 is constant and β_2 depends only on $|z|$.

Using 1), 2) and Theorem 1, we obtain without difficulties the following analog of Dini's Theorem:

Theorem 6 *(Generalization of Dini's Theorem to the case of an isolated non-regular point) Two Riemannian metrics G_1 and G_2 on a surface M, which are proportional in an isolated point q_0, are geodesic equivalent in a neighborhood of this point if and only if one of the following two conditions holds:*

1. *In a neighborhood of q_0, there exist coordinates (x, y), $q_0 = (0, 0)$ and two one-variable smooth functions U and V, satisfying $0 < U(u) < V(0) = U(0) < V(v)$ for all positive u and v, $U'(0) = -V'(0)$, and $V'(0) > 0$, such that in the punctured neighborhood of q_0 the metrics G_1 and G_2 satisfy*

$$||\cdot||_1^2 = \left(\frac{1}{U\left(r\cos^2\frac{\theta}{2}\right)} - \frac{1}{V\left(r\sin^2\frac{\theta}{2}\right)}\right)\frac{1}{4r}(dr^2 + r^2 d\theta^2), \tag{5.7}$$

$$||\cdot||_2^2 = \frac{S}{8r}((A - S\cos\theta)dr^2 - 2Sr\sin\theta dr d\theta + (A + S\cos\theta)r^2 d\theta^2), \tag{5.8}$$

where

$$A = U\left(r\cos^2\frac{\theta}{2}\right) + V\left(r\sin^2\frac{\theta}{2}\right), \quad S = V\left(r\sin^2\frac{\theta}{2}\right) - U\left(r\cos^2\frac{\theta}{2}\right),$$

and (r, θ) are the corresponding polar coordinates;

2. *In a neighborhood of* q_0*, there exist coordinates* (x, y)*,* $q_0 = (0, 0)$*, positive constants* a*,* C*, and an one-variable smooth functions* $R(r)$*, satisfying* $R(r) \neq R(0)$ *for* $r > 0$*,* $R(0) = C$*,* $R'(0) = 0$*, and* $R''(0) \neq 0$*, such that in the punctured neighborhood of* q_0 *the metrics* G_1 *and* G_2 *satisfy*

$$||\cdot||_1^2 = \left|\frac{1}{C} - \frac{1}{R(r)}\right| \frac{a}{r^2}(dr^2 + r^2 d\theta^2), \tag{5.9}$$

$$||\cdot||_2^2 = \frac{aCR(r)}{r^2}\left|\frac{1}{C} - \frac{1}{R(r)}\right| \left(R(r)dr^2 + Cr^2 d\theta^2\right), \tag{5.10}$$

where (r, θ) *are the corresponding polar coordinates.*

Remark 4 *The conditions on the functions* U *and* V *in the case 1 and on* R *in the case 2 follows easily from the fact that the metrics are positive definite and nonsingular at* q_0*.*

Remark 5 *Using the standard arguments of Complex Analysis, one can show that for the pair of geodesically equivalent metrics the set of non-regular points cannot be a rectifiable curve* Γ*: in this case one can construct an one-valued Dini transition conformal function out of* Γ *which goes to infinity, when one tends to* Γ*. Then by Morera Theorem the function* $1/F$ *is analytic and equal to zero on* Γ *and so everywhere, which is impossible.*

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