Injectivity properties of pole placement maps of linear control systems

Igor Zelenko

Based on joint work with Frank Sottile and Yanhe Huang

Texas A&M University, USA

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Pole placement map

$\Sigma = (A, B, C)$, where A, B, C are complex matrices of sizes $N \times N$, $N \times m$ and $p \times N$ such that the linear control system

$$\dot{x} = Ax + Bu,$$
$$y = Cx$$
$$x \in X = \mathbb{C}^N, y \in Y = \mathbb{C}^p, u \in U = \mathbb{C}^m$$

is controllable and observable.

Transfer function $G(s) = C(sI - A)^{-1}B$.

Feedback $u = Ky$, where K is a $m \times p$ matrix K, $\rightarrow$ closed loop system $\dot{x} = (A + BKC)x$.

Pole placement map $F_\Sigma : \mathrm{Mat}_{m \times p} \rightarrow \mathbb{C}_N[s]$,

$$F(K)(s) = \det(sI - A - BKC).$$

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Statement of the problem

We assume that $N > mp$, so F is not onto (i.e. an arbitrary configuration of poles is not realizable).

Question *Under what condition on the control system does the general polynomial in the image of F has at least two preimage (or , equivalently, general realizable configuration of poles is realized at least by two feedbacks).*

Obvious examples:

- (Symmetric systems or state-feedback equivalent to them) $A = A^T$, $C = B^T \Leftrightarrow G(s)$ is symmetric. Then $F(K) = F(K^T)$;
- (Skew-symmetric systems or state-feedback equivalent to them) N is even and for some J such that $J^T = -J$ and $J^2 = -I$, we have $(AJ)^T = -AJ$, $C = -B^T J \Leftrightarrow G(s)$ is skew-symmetric.Then $F(K) = F(-K^T)$;

Are these the only examples in the case $N > mp$ when the degree of F is greater than 1?

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Extending the pole placement map to the Grassmannian

The map $K \in \mathrm{Hom}(Y, U) \mapsto \mathrm{Graph}\, K$ is the bijection onto the affine coordinate domain $(0 \times U)^{\pitchfork}$ of $\mathrm{Gr}_p(Y \times U)$ consisting of all p-dimensional subspaces transversal to $0 \times U$. Hence, the map F is well defined on the affine coordinate domain of $\mathrm{Gr}_p(Y \times U)$: $F(\mathrm{Graph}\, K) := F(K)$. It can be extended to the whole

Grassmannian: Use the coprime factorization of the transfer function $G(s)$, $G(s) = C(sI - A)^{-1}B = E(s)D(s)^{-1}, \quad \det D(s) = \det(sI - A)$.

Then $F(\mathrm{Graph}\, K)(s) = F(K)(s) = \det \begin{pmatrix} D(s) & K \\ E(s) & I_p \end{pmatrix}$

and the extension to $\mathrm{Gr}_p(Y \times U)$ is given by

$$F(L) = \left[\det \begin{pmatrix} D(s) & K_1 \\ E(s) & K_2 \end{pmatrix} \right], \tag{1}$$

where $L \in \mathrm{Gr}_p(Y \times U)$ is spanned by the last p columns of the matrix in (1) and $[\cdot]$ is an equivalence class in the projective space.

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More general point of view: central projections of Grassmannian

Let V be a complex vector space ($\dim V = m + p$) and $\wedge^p V$ be the pth alternating tensor power of V.

Plücker embedding $\mathrm{Pl} : \mathrm{Gr}_p(V) \to \mathbb{P}(\wedge^p V)$:

$$\mathrm{span}(v_1, \ldots, v_p) \to v_1 \wedge v_2 \ldots \wedge v_p.$$

The image of Pl will be called the Grassmann variety and it will be also denoted by $\mathrm{Gr}_p(V)$.
Given a subspace $X \subset \wedge^p V$, let $\hat{\pi}_X : \wedge^p V \to (\wedge^p V)/X$ be the canonical projection.
This induces a map $\pi_X : \mathbb{P} \wedge^p V \to \mathbb{P}(\wedge^p V/X) \cup \{0\}$ (here $\pi_X^{-1}(\{0\}) = \mathbb{P}X$).
Restrict π_X to $\mathrm{Gr}_p(V)$-the central (or linear) projection of $\mathrm{Gr}_p(X)$ by X.
We are interested in the question *when the degree of this restriction is finite and greater than* 1*?*

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More general point of view: central projections of Grassmannian

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Pole placement map via a central projection

$$F(L) = \left[\det \begin{pmatrix} D(s) & K_1 \\ E(s) & K_2 \end{pmatrix}\right],$$

where $L \in \mathrm{Gr}_p(Y \times U)$ is spanned by the last p columns of the matrix.

Taking the span of the first m columns of the same matrix at each $s \in \mathbb{C}$, we get a curve $s \mapsto \Gamma(s)$, the Hermann-Martin curve of our control system, in $\mathrm{Gr}_m(Y \times U)$. The transfer function $G(s)$ is the coordinate representation of the Hermann-Martin curve in an affine chart of $\mathrm{Gr}_m(Y \times U)$.

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Some general facts on central projections

Let $X \subset \wedge^p V$, $\dim V = m + p$.

If $\operatorname{codim} X = \dim \mathrm{Gr}_p(V) + 1 = mp + 1$ and $\mathbb{P}X \cap \mathrm{Gr}_p(V) = \emptyset$, then the map π_X is finite and the degree of the map π_X is equal to $\frac{1!2!\ldots(p-1)!\cdot(mp)!}{m!(m+1)!\ldots(m+p-1)!}$ (Schubert, 1886)

If $\operatorname{codim} X > mp + 1$, then for generic X the degree of the map π_X is equal to 1.

For which X with $\operatorname{codim} X > mp + 1$ the degree of the map π_X is finite and greater than 1?

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Central projections induced by finite order linear maps

First, we want to characterize all $X \subset \wedge^p V$ such that there exists a nontrivial finite order linear automorphism $\widehat{A}$ of $\wedge^p V$ with the induced automorphism A of the projective space $\mathbb{P} \wedge^p V$ satisfying

1. A preserves the Grassmannian $\mathrm{Gr}_p(V)$, i.e. $A\big(\mathrm{Gr}_p(V)\big) \subset \mathrm{Gr}_p(V)$;
2. A preserves the fibers of the map π_X, i.e. $\pi_X(Ay) = \pi_X(y)$ for all $y \in \mathrm{Gr}_p(V) \Rightarrow \deg \pi_X > 1$.

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Central projections induced by linear automorphisms (continued)

Proposition (F. Sottile, Y. Huang, I.Z)

If $X \subset \wedge^p V$ is induced by a finite order linear automorphism $\widehat{A}$, then X contains all eigenspaces of A except one.

Theorem (Wei-Liang Chow 1949)

Consider an automorphism $\widehat{A}$ of $\wedge^p V$ such that the corresponding automorphism A of the projective space $\mathbb{P} \wedge^p V$ preserves the Grassmannian $G_p(V)$. Then

1. *either A is induced by a linear automorphism of V,*
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What special in a Lagrangian involution?

If $X \subset \wedge^p V$ is induced by a finite order linear automorphism of $\wedge^p V$ of Chow's type 2, then X is also induced by order 2 linear automorphism of Chow's type 2 such that the corresponding bilinear form is either symmetric or skew-symmetric (symplectic).

Note that the pole placement map for a *symmetric control systems correspond to the case of symplectic form* and for a *skew-symmetric control system corresponds to a symmetric form*.

Theorem (F. Sottile, Y. Huang, I.Z.)

If $\mathbb{P}X \cap \mathrm{Gr}_p(V) = \emptyset$ and X is induced by a nontrivial linear automorphism of $\wedge^p V$, then $p = m$ and X is induced by a linear automorphism of Chow's type 2 corresponding to a symplectic form on V (i.e., to a Lagrangian involution).

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If $\mathbb{P}X \cap \mathrm{Gr}_p(V) = \emptyset$, $\operatorname{codim} X > mp + 1$, *and the degree of* π_X *is* 2, *then* X *is induced by a Lagrangian involution with respect to some symplectic form* ω *on* V.

Theorem (F. Sottile, Y. Huang, I.Z.)

1. *If* $m = p = 2$ *and* $\mathbb{P}X \cap \mathrm{Gr}_2(V) = \emptyset$, *then the degree of* π_X *is greater than* 1 *if and only if* X *is induced by a Lagrangian involution with respect to some symplectic form* ω *on* V;
2. *If* $m = p = 3$, $\mathbb{P}X \cap \mathrm{Gr}_3(V) = \emptyset$, *and* $\dim X \leq 5$, *then the degree of* π_X *is equal to* 1;
3. *If* $m = p = 3$, $\mathbb{P}X \cap \mathrm{Gr}_3(V) = \emptyset$, *and* $\dim X = 6$ *(which is the minimal possible dimension of* X *induced by a Lagrangian involution), then the degree of* π_X *is greater than* 1 *if and only if* X *is induced by a Lagrangian involution with respect to some symplectic form* ω *on* V.

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What is special in Lagrangian involution (continued)

Theorem (F. Sottile, Y. Huang, I.Z.)

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Applications to pole placement map

Consider the linear control system Σ as before , $V = (Y \times U)^*$.

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If $X_\Sigma \cap \mathrm{Gr}_m(V) = \emptyset$, $\mathrm{codim}\, X_\Sigma > mp + 1$, and the degree of the pole placement map is 2, then the control system is state-feedback equivalent to a symmetric control system.

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Another application: Wronski map

$$\Lambda = \operatorname{span}\{f_1(t), f_2(t), \ldots, f_p(t)\}, \quad \dim \Lambda = p$$

$$\operatorname{Wr}\big(f_1(t), f_2(t), \ldots, f_p(t)\big) := \det \begin{pmatrix} f_1(t) & f_2(t) & \ldots & f_p(t) \\ f_1'(t) & f_2'(t) & \ldots & f_p'(t) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(p-1)}(t) & f_2^{(p-1)}(t) & \ldots & f_p^{(p-1)}(t) \end{pmatrix}.$$

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Wronski map (continued)

Consider Linear differential operator

$$Lx = x^{(m+p)}(t) + a_{m+p-1}(t)x^{(m+p-1)}(t) + \ldots + a_0(t)x(t)$$

Let V_L be the space of solution of $Lx = 0$.

$$\mathrm{Wr} : \; \mathrm{Gr}_m(V_L) \longrightarrow \mathbb{P}(\mathcal{C}^\infty)\,.$$

1. Wronki map is also equivalent to certain central projection π_{X_L} of $\mathrm{Gr}_p(V_L)$ for some $X_L \subset \wedge^p V_L$.
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Thanks for your attention.