ON LOCAL GEOMETRY OF RANK 3 DISTRIBUTIONS WITH 6-DIMENSIONAL SQUARE

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Abstract. We solve the equivalence problem for rank 3 completely nonholonomic vector distributions with 6-dimensional square on a smooth manifold of arbitrary dimension n under very mild genericity conditions. The main idea is to consider the projectivization of the annihilator $D^\perp$ of a given 3-dimensional distribution D. It is naturally foliated by characteristic curves, which are also called the abnormal extremals of the distribution D. The dynamics of vertical fibers along characteristic curves defines certain curves of flags of isotropic and coisotropic subspaces in a linear symplectic space. The problem of equivalence of distributions can be essentially reduced to the differential geometry of such curves.

The class of all 3-distributions under consideration is split into a finite number of subclasses according to the Young diagram of their flags. The local geometry of distributions can be recovered from the properties of the symmetry group of so-called flat curves of flags associated with this Young diagram. In each subclass we describe the flat distribution and construct a canonical frame for any other distribution.

It turns out that for $n > 6$ in the most nontrivial case the symmetry algebra of the flat distribution can be described in terms of rational normal curves (their secants and tangential developables) in projective spaces and its dimension grows exponentially with respect to n.

1. Introduction

This paper is a next step of the long-standing program of studying the geometry of nonholonomic vector distributions using the ideas of geometric control theory. In earlier articles [3, 4] we have solved the equivalence problem for rank 2 vector distributions constructing a canonical frame under very mild genericity conditions.

In this article we treat rank 3 vector distributions on smooth manifolds of arbitrary dimension and solve the equivalence problem constructing a canonical frame for each three-dimensional distribution D satisfying certain non-degeneracy conditions. The first such condition is the assumption that the dimension of the derived distribution $D^2 = D + [D, D]$ is 6. In the following we shall refer to such distributions as $(3, 6, \dots)$-distributions indicating that the dimensions of D itself and D^2 are 3 and 6 respectively.

The case of $(3, 6)$-distributions on 6 dimensional manifolds was considered by Robert Bryant [1] using the Cartan equivalence method. In particular, Bryant proves that there is a natural parabolic geometry of type B_3 associated to each such distribution. The aim of the current article is to construct similar geometries associated with a given $(3, 6, \dots)$-distribution in any dimension. As in the case of rank 2 vector distributions the structure groups we get in dimensions 7 and higher are no longer semisimple.

The obvious (but very rough in the most cases) discrete invariant of a distribution D at q a so-called *small growth vectors* at q. It is the tuple $\{\dim D^j(q)\}_{j\in\mathbb{N}}$, where D^j is the j-th power

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of the distribution D, i.e., $D^j = D^{j-1} + [D, D^{j-1}]$, $D^1 = D$. More generally, at each point $q \in M$ we can consider the graded space $\mathfrak{m}_q = \sum_{i \geq 1} D^{i+1}/D^i$. It can be naturally equipped with a structure of a graded nilpotent Lie algebra and is called a symbol of the distribution D at a point q. The notion of this symbol is extensively used in works of N. Tanaka and his school (see [10]) who systematized and generalized the Cartan equivalence method. However, these tools become really effective only when the symbol algebras are isomorphic at different points, and all constructions strongly depend on the algebraic structure of the symbol. Note that the problem of classification of all symbols (graded nilpotent Lie algebras) is quite nontrivial already in dimension 7 (see [8]) and it looks completely hopeless for arbitrary dimensions. For example, as was shown in [8] already in dimension 7 the continuous parameters appears in symbols of rank (3,6,...)-distributions (see models $m7\_3\_3(\alpha)$ and $m7\_3\_13(\alpha)$ there), and there are 6 more non-isomorphic symbols in addition to that.

The core idea of our approach comes from geometric control theory and is based on construction of a characteristic line bundle associated with a given $(3, 6, \dots)$-distribution and the study of curves of flags of (co)isotropic subspaces obtained by its linearization. Our classification of rank 3 distributions is done according to a so-called Young diagram of these curves of flags and is not directly related to Tanaka symbols of the distribution D itself. The local geometry of distributions can be recovered from the properties of symmetry groups of so-called flat curves of flags associated with a given Young diagram.

Below we outline the main constructions and the results of the paper. They are given without proofs and repeated in more detail and with proofs in the main part of the paper.

1.1. **Characteristic 1-foliation of abnormal extremals.** First we distinguish a characteristic 1-foliation (the foliation of abnormal extremals) on a special odd-dimensional submanifold of the cotangent bundle associated with any rank 3 distribution D. Define the j-th power of the distribution D as $D^1 = D$ and $D^{i+1} = D^i + [D, D^i]$. We assume that all D^j are subbundles of the tangent bundle. Denote by $(D^j)^\perp \subset T^*M$ the annihilator of the j-th power D^j, namely

$$(D^j)^\perp = \{(p, q) \in T^*_q M \mid p \cdot v = 0 \quad \forall\, v \in D^j(q)\}.$$

Let $\pi \colon T^*M \mapsto M$ be the canonical projection. For any $\lambda \in T^*M$, $\lambda = (p, q)$, $q \in M$, $p \in T^*_q M$, let $\varsigma(\lambda)(\cdot) = p(\pi_* \cdot)$ be the canonical Liouville form and $\hat{\sigma} = d\varsigma$ be the standard symplectic structure on T^*M.

The crucial notion in this paper is *an abnormal extremal of a distribution.* An *unparametrized* curve in $D^\perp$ is called *abnormal extremal of a distribution* D if the tangent line to it at almost every point belongs to the kernel of the restriction $\hat{\sigma}|_{D^\perp}$ of $\hat{\sigma}$ to $D^\perp$ at this point. The term “abnormal extremals” comes from Optimal Control Theory: abnormal extremals of D are exactly Pontryagin extremals with zero Lagrange multiplier near the functional for any variational problem with constrains, given by the distribution D.

Since $\dim D = 3$, the submanifold $D^\perp$ has odd codimension in T^*M, and the kernels of the restriction $\hat{\sigma}|_{D^\perp}$ are non-trivial. Moreover, as we show below (Lemma 1), for points in $D^\perp \backslash (D^2)^\perp$ these kernels are one-dimensional. They form a *characteristic line distribution* in $D^\perp \backslash (D^2)^\perp$, which will be denoted by $\widehat{\mathcal{C}}$. The line distribution $\widehat{\mathcal{C}}$ defines in turn a *characteristic 1-foliation* on $D^\perp \backslash (D^2)^\perp$. The leaves of this foliation are exactly the abnormal extremals of the distribution D lying in the complement to $(D^2)^\perp$.

It is more natural to work on the projectivization of the contangent bundle instead of the tangent bundle itself. We define the same objects on the projectivization of $D^\perp \backslash (D^2)^\perp$. As homotheties of the fibers of $D^\perp$ preserve the characteristic line distribution, the projectivization induces the *characteristic line distribution* on $\mathbb{P}D^\perp\backslash\mathbb{P}(D^2)^\perp$, which will be denoted by $\mathcal{C}$. It defines *the characteristic* 1*-foliation*, and its leaves are called the *abnormal extremals of the distribution* D on $\mathbb{P}D^\perp\backslash\mathbb{P}(D^2)^\perp$.

The distribution $\mathcal{C}$ can be defined equivalently in the following way. Take the corank 1 distribution on $D^\perp\backslash(D^2)^\perp$, given by the Pfaffian equation $\varsigma|_{D^\perp} = 0$ and push it forward it under projectivization to $\mathbb{P}D^\perp$. In this way we obtain a corank 1 distribution on $\mathbb{P}D^\perp$, which will be denoted by $\widetilde{\Delta}$. The distribution $\widetilde{\Delta}$ defines a quasi-contact structure on the even dimensional manifold $\mathbb{P}D^\perp\backslash\mathbb{P}(D^2)^\perp$ and $\mathcal{C}$ is exactly the characteristic distribution of this quasi-contact structure. Moreover, the symplectic form $\hat{\sigma}$ induces the antisymmetric form on each subspace of a distribution $\widetilde{\Delta}$, defined up to a multiplication by a constant. This antisymmetric form will be denoted by $\tilde{\sigma}$.

1.2. **Lifting of the distribution to the cotangent bundle.** We can consider the characteristic distribution $\mathcal{C}$ as a dynamical system naturally associated with any $(3,6,\dots)$-distribution and study the dynamics of fibers of the natural projection $\pi\colon \mathbb{P}D^\perp\backslash\mathbb{P}(D^2)^\perp \to M$.

In more detail, let V be a vertical distribution defined as a set of tangent spaces to the fibers of the projection π. Then, clearly, V is complimentary to $\mathcal{C}$, and we arrive at a so-called *pseudo-product structure*, which consists of a pair of 2 completely integrable distributions $(\mathcal{C}, V)$, whose sum is non-integrable. Indeed, the direct computation shows that the pull-back π^*D of D itself is easily recovered from $\mathcal{C}$ and V. Namely, we have $\pi^*D = V + \mathcal{C} + [\mathcal{C}, V]$. In particular, this proves that the distribution $\mathcal{C}\oplus V$ is bracket-generating.

In this paper we require a stronger non-degeneracy condition on the distribution D. We construct a sequence of distributions starting from $\mathcal{J}^{(-1)} = \mathcal{C}\oplus V$ by taking the brackets only with the sections of characteristic bundle:

$$\begin{aligned}\mathcal{J}^{(-1)} &= \mathcal{C}\oplus V;\\ \mathcal{J}^{(0)} &= \mathcal{J}^{(-1)} + [\mathcal{C}, \mathcal{J}^{(-1)}] \quad (= \pi^*D);\\ \mathcal{J}^{(i+1)} &= \mathcal{J}^{(i)} + [\mathcal{C}, \mathcal{J}^{(i)}], \quad i \geq 0.\end{aligned}$$

As $\mathcal{J}^{(-1)}$ lies in the restriction of the quasi-contact bundle $\widetilde{\Delta}$ to $\mathbb{P}D^\perp\backslash\mathbb{P}(D^2)^\perp$ and $\mathcal{C}$ is the characteristic bundle of this restriction, it is clear that all distributions $\mathcal{J}^{(i)}$ will be subbundles of the contact bundle. We say that the distribution D is *of maximal class* if $\mathcal{J}^{(m)} = \widetilde{\Delta}$ for sufficiently large m.

In this paper we consider only $(3,6,\dots)$-distributions of maximal class. We note that we are not aware of any examples of $(3,6,\dots)$-distributions that are not of maximal class.

We can prolong the sequence of subbundles $\mathcal{J}^{(i)}$ to the negative side defining:

$$\mathcal{J}^{(i-1)} = \{X \in \mathcal{J}^{(i)} \mid [X, \mathcal{C}] \subset \mathcal{J}^{(i)}\}.$$

It appears that the subspace $\mathcal{J}^{(-i-1)}$ is exactly the skew-orthogonal complement to $\mathcal{J}^{(i)}$ with respect to the form $\tilde{\sigma}$ for any $i \geq 0$. In other words, we have:

$$\mathcal{J}^{(-i-1)} = \{X \in \widetilde{\Delta} \mid \tilde{\sigma}(X, \mathcal{J}^{(i)}) = 0\}.$$

Thus, we immediately see that the sequence $\mathcal{J}^{(-i)}$ descends to the characteristic bundle $\mathcal{C}$ for sufficiently small i. In addition, using the fact that $\dim \mathcal{J}^{(0)} - \dim \mathcal{J}^{(-1)} = 2$, we prove that $\dim \mathcal{J}^{(i)} - \dim \mathcal{J}^{(i-1)} \leq 2$ for any $i \in \mathbb{Z}$.

Thus, at any generic point $\lambda \in \mathbb{P}D^\perp\backslash\mathbb{P}(D^2)^\perp$ we have a flag of subbundles:

$$0 \subset \mathcal{C} = \mathcal{J}^{(-m-1)} \subset \mathcal{J}^{(-m)} \subset \dots \subset \mathcal{J}^{(-1)} = \mathcal{C} + V \subset \mathcal{J}^{(0)} = \pi^*D \subset \mathcal{J}^{(1)} \subset \dots \subset \mathcal{J}^{(m)} = \widetilde{\Delta},$$

where the dimension gap between two neighbors in this sequence is either 2 or 1.

1.3. **Linearization of the flag along characteristic foliation.** We *linearize* this sequence along the characteristic foliation and turn it into the curve in an appropriate flag manifold of a symplectic space at each point $\lambda_0 \in \mathbb{P}D^\perp \backslash \mathbb{P}(D^2)^\perp$.

Namely, let γ be a leaf of the characteristic 1-foliation $\mathcal{C}$ containing λ_0 and let N be a manifold of all leaves of $\mathcal{C}$ in a small neighborhood of λ_0. Then γ represents a point of N, and we have a natural projection $\Phi\colon \mathbb{P}D^\perp \backslash \mathbb{P}(D^2)^\perp \to N$ defined in a neighborhood of λ_0. Let $\Delta \subset T_\gamma N$ be the image of $\widetilde{\Delta}$ under this projection. As it is a quotient of a codimension 1 distribution by its characteristic, it inherits a symplectic structure σ defined up to a multiplication by a non-zero scalar.

The differential Φ_* takes $\mathcal{C}_\lambda$ to 0 and the flag of spaces $\{\mathcal{J}^{(i)}\}$ at a point λ to a certain flag of subspaces in Δ:

$$\lambda \mapsto \left\{0 = J^{(-m-1)} \subset J^{(-m)} \subset \cdots \subset J^{(m-1)} \subset J^{(m)} = \Delta\right\}, \quad \lambda \in \gamma,$$

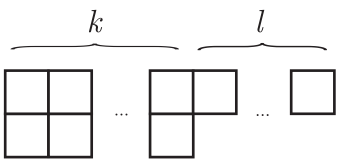
where $J^{(i)} = \Phi_*(\mathcal{J}^{(i)})$ for all $i \in \mathbb{Z}$.

Thus, we get a natural map from γ to the variety of all flags in the symplectic space Δ. We call this curve *the linearization* of the flag $\{\mathcal{J}^{(i)}\}$ along the characteristic curve γ. As it is defined in a natural way, any invariants of this curve will be automatically the invariants of the distribution D itself. One of the main points here is that the local geometry of rank 3 distribution can be reconstructed from the geometry of such curves of flag. Hence the core part of the paper is devoted to the study of the geometry of these curves.

1.4. **Geometry of curves of flags of (co)isotropic subspaces.**

(a) Young diagrams. To any curve of flags $\{J^{(i)}\}$ above one can construct a Young diagram: the number of boxes in i-th row of it is equal to $\dim J^{(i)} - \dim J^{(i-1)}$. As the negative part of the flag is a skew-symmetric complement to the non-negative part, this diagram completely determines the dimensions of all quotient spaces $J^{(i)}/J^{(i-1)}$ for any $i \in \mathbb{Z}$.

By construction there are no more than 2 boxes in each column. So, the diagram has the form:



and is completely determined by a pair of integers (k, l). This diagram is said to be of type (k, l). It is easy to get that k and l have to satisfy the relation $n = 2k + l + 2$, where n is the dimension of the base manifold M. In particular, the parity of l and n should coincide. Besides, the assumption $\dim D^2 = 6$ implies that the number of columns with 2 boxes is not less than 2. The case $l = 0$ corresponds to *rectangular* Young diagrams, while $l > 0$ corresponds to *non-rectangular* Young diagrams. We say that a $(3, 6, \dots)$-distributions of maximal class is of the type (k, l) if germs of linearization of the flag $\{\mathcal{J}^{(i)}\}$ at generic points of $\mathbb{P}D^\perp \backslash \mathbb{P}(D^2)^\perp$ (along the corresponding characteristic curve) have type (k, l).

(b) Flat curves associated with Young diagrams. All curves of flags with given Young diagram can be treated uniformly as a deformation of the so-called *flat curve* having the biggest group of symmetries. To describe the flat curve let V be a vector space of the same dimension as $\Delta(\gamma)$ (equal to $2n - 6 = 4k + l - 2$) endowed with a one-parametric family of symplectic forms such that any form from this family is obtained from any other by a multiplication on a nonzero constant. Assume also that V is endowed with a filtration

$$V = V^{(k+l-1)} \supset \ldots \supset V^{(-k-l+1)} \supset V^{(-k-l)} = 0 \tag{1.1}$$

such that $\dim V^{(i)} = \dim J^{(i)}$, $V^{(i)}$ are isotropic for $i < 0$ and $V^{(1-i)}$ is the skew symmetric complement of $V^{(i)}$ for any i. Finally assume that V is endowed with a distinguished basis $(e_1, \ldots, e_{2k+l-1}, f_1, \ldots, f_{2k+l-1})$ such that

(1) this basis is symplectic w.r.t. to one of the form σ from the one-parametric family of symplectic forms on V, i.e. $\sigma(e_i, e_j) = \sigma(f_i, f_j) = 0$, for any i, j, $\sigma(e_i, f_{2k+l-i}) = (-1)^i$, and $\sigma(e_i, f_j) = 0$ for any i, j such that $i + j \neq 2k + l$;
(2) the filtration (1.1) coincides with

$$\begin{aligned} 0 \subset \langle e_1\rangle \subset \langle e_1, e_2\rangle \subset \cdots \subset \langle e_1, \ldots, e_l\rangle \\ \subset \langle e_1, \ldots, e_{l+1}, f_1\rangle \subset \langle e_1, \ldots, e_{l+2}, f_1, f_2\rangle \subset \cdots \subset \langle e_1, \ldots, e_{2k+l-1}, f_1 \ldots, f_{2k-1}\rangle \\ \subset \langle e_1, \ldots, e_{2k+l-1}, f_1, \ldots, f_{2k}\rangle \subset \cdots \subset \langle e_1, \ldots, e_{2k+l-1}, f_1, \ldots, f_{2k+l-1}\rangle = V. \end{aligned} \tag{1.2}$$

Now define a linear maps $X \in \mathrm{End}(V)$ as follows:

$$Xe_i = e_{i+1}, \quad Xf_i = f_{i+1} \text{ for } i = 1, \ldots 2k + l - 2, \quad Xe_{2k+l-1} = Xf_{2k+l-1} = 0.$$

We say that the curve of flags $\mathfrak{F}_{k,l} = \{\mathfrak{F}^{(i)}_{k,l}\}_{i=-k-l}^{k+l-1}$ is a *flat curve associated with the Young diagram of type* (k, l), if it is an orbit of the flag (1.2) under the action of the one-parameter subgroup $\exp(tX)$. A symplectic moving frame $(\tilde e_1(\cdot), \ldots, \tilde e_{2k+l-1}(\cdot), \tilde f_1(\cdot), \ldots, \tilde f_{2k+l-1}(\cdot))$ is called *normal moving frame of the flat curve* $\mathfrak{F}_{k,l}$, if

$$\begin{aligned} &\mathfrak{F}^{(i)}_{k,l}(\cdot) = \langle \tilde e_1(\cdot), \ldots \tilde e_{i+k+l}(\cdot)\rangle, \quad i = -k - l, \ldots, -k \\ &\mathfrak{F}^{(i)}_{k,l}(\cdot) = \langle \tilde e_1(\cdot), \ldots \tilde e_{i+k+l}(\cdot), \tilde f_1(\cdot), \ldots, \tilde f_{i+k}(\cdot)\rangle, \quad i = -k + 1, \ldots, k + l - 1 \end{aligned}$$

and for some parametrization t of the curve

$$\tilde e_i'(t) = \tilde e_{i+1}(t), \tilde f_i'(t) = \tilde f_{i+1}(t), \text{ for } i = 1, \ldots 2k + l - 2, \quad \tilde e_{2k+l-1}'(t) = \tilde f_{2k+l-1}'(t) = 0.$$

Note that the frame $(\exp(tX)e_1, \ldots, \exp(tX)e_{2k+l-1}, \exp(tX)f_1, \ldots, \exp(tX)f_{2k+l-1})$, where e_i and f_i are as above, is a normal moving frame of the flat curve $\{\mathfrak{F}^{(i)}\}_{i=-k-l}^{k+l-1}$.

Let $\mathfrak{S}_{k,l}$ be the group of all isomorphisms A of V, sending the flat curve to itself and preserving the one-parametric family of symplectic forms on V. The latter means that for any form σ from this family there exists a nonzero constant c such that

$$\sigma(Av_1, Av_2) = c\sigma(v_1, v_2), \quad \forall v_1, v_2 \in V. \tag{1.3}$$

In other words, $\mathfrak{S}_{k,l}$ is the group of symmetries of the flat curve. Denote $\mathfrak{s}_{k,l}$ the corresponding Lie algebras (i.e. the Lie algebra of infinitesimal symmetries of the flat curve).

(c) Bundles of moving frames and their symbol.

Let us clarify what do we mean by saying that any curve of flags with given Young diagram is a deformation of the corresponding flat curve. For any such curve we construct a bundle of canonical moving frames of dimension equal to the dimension of the group of symmetries $\mathfrak{S}_{k,l}$ of the flat curve. For the flat curve this bundle of canonical moving frames coincides with all its normal moving frames defined above. Take as before a manifold N of all leaves of the characteristic foliation $\mathcal{C}$ in a small neighborhood of a point $\lambda_0 \in \mathbb{P}D^\perp \backslash \mathbb{P}(D^2)^\perp$ such that the linearizations of the flag $\{\mathcal{J}^{(i)}\}$ along any leaf of $\mathcal{C}$ in this neighborhood has the Young diagram of type (k, l). Take $\gamma \in N$ and collect all frames on $\Delta(\gamma)$, obtained from all canonical moving frames for the linearization of of the flag $\{\mathcal{J}^{(i)}\}$ along γ. In this way we get the canonical frame bundle P

on the contact distribution Δ of the manifold N. By the frame bundle on a distribution of a manifold we mean a fiber bundle over this manifold with the fiber over a point consisting of some distinguished frames of the distribution at this point. The frame bundle we construct is not in general a principle fiber bundle, but it still possesses a number of nice properties such as a *constant symbol*.

In more detail, we define a symbol of the frame bundle P on Δ as follows. Take the subspace V with the distinguished frame as in paragraph (b). Then any frame p on Δ can be identified with the isomorphism between V and Δ, sending the distinguished frame on V to the frame p. Hence, the tangent space to a fiber of P at a point p can be identified with a subspace of $\mathfrak{gl}(V)$. The filtration on V induces a natural filtration on $\mathfrak{gl}(V)$ and, therefore, on any its subspace. The corresponding graded subspace $\operatorname{gr} T_pP$ is called a *symbol of the bundle P at a point p*. Symbols are subspaces of $\operatorname{gr}\mathfrak{gl}(V)$, which, in turn, is naturally identified with $\mathfrak{gl}(\operatorname{gr} V)$. In the case of rectangular diagram already the tangent spaces to a fiber of P at different points are the same, as subspaces of $\mathfrak{gl}(V)$. On the other hand, in the case of nonrectangular diagram this tangent spaces at different points are different subspaces of $\mathfrak{gl}(V)$. However, all symbols at different points of this bundle are the same. Moreover, in both cases the symbol is equal to the algebra of infinitesimal symmetries $\mathfrak{s}_{k,l}$ of the flat curve (under the natural identification of V and $\operatorname{gr} V$ via the distinguished basis on V).

(d) Prolongation procedure. We emphasize that our frame bundles on the corank 1 distribution Δ are not even principle bundles in general, but the additional filtration on the model space V for these bundles allows to define the notion of symbol of the bundle at a point. It turns out that assuming that the symbol is constant, it is possible to carry prolongation procedure for such frame bundles in a similar way as in the classical theory of G-structures on manifold and as in the Tanaka theory for G-structures on filtered manifolds [10]. Since originally we have frames not on the whole tangent bundle but on a corank 1 distribution only, we have to modify the notion of the Spencer operator and of the prolongation (of subspaces of $\mathfrak{gl}(V)$ and of frame bundles). As in case of standard G-structures, if for some $i > 0$ the modified i-th prolongation of the symbol $\mathfrak{s}_{k,l}$ of our frame bundle P is trivial, then for any distribution of type (k, l) there exists a canonical frame on a certain bundle Q over N. This bundle is constructed as the i-th prolongation of the frame bundle P.

The modified i-th prolongation $\mathfrak{s}_{k,l}^{(im)}$ of the symbol $\mathfrak{s}_{k,l}$ has a simple description in terms of the Tanaka prolongation of a certain graded Lie algebra. First let $\mathfrak{g}_{-1} = V$ and $\mathfrak{g}_{-2} = \mathbb{R}\eta$ for some vector η. Define the structure of a graded Lie algebra on the vector space $\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ by setting $[v_1, v_2] = \sigma(v_1, v_2)\eta$ for some form σ from the one-parametric family of symplectic forms on V. This Lie algebra is isomorphic to the Heisenberg algebra. Now set $\mathfrak{g}_0 = \mathfrak{s}_{k,l}$. Note that by construction $\mathfrak{s}_{k,l}$ is a subalgebra of the Lie algebra $\mathfrak{csp}(V)$ corresponding to a Lie group of all isomorphisms A of V satisfying (1.3). On the other hand, it is clear that $\mathfrak{csp}(V)$ coincides with the algebra of all derivations of $\mathfrak{g}_{-2}\oplus\mathfrak{g}_{-1}$ preserving the filtration. Therefore the space $\mathfrak{g}_{-2}\oplus\mathfrak{g}_{-1}\oplus\mathfrak{g}_0$ can be endowed with the natural structure of a graded Lie algebra as well by setting $[A, v] = Av$ for any $A \in \mathfrak{g}_0$ and $v \in V$. Let

$$\mathfrak{G}_{k,l} = \bigoplus_{i\geq -2} \mathfrak{g}_i = \mathbb{R}\eta \oplus V \oplus \mathfrak{s}_{k,l} \oplus \bigoplus_{i\geq 1} \mathfrak{g}_i$$

be the Tanaka universal prolongation ([10]) of the graded Lie algebra

$$\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 = \mathbb{R}\eta \oplus V \oplus \mathfrak{s}_{k,l}. \tag{1.4}$$

It turns out that *our modified ith prolongation* $\mathfrak{s}_{k,l}^{(im)}$ *coincides with the Tanaka ith prolongation* $\mathfrak{g}_i$ *of the algebra* $\mathfrak{g}_{-2}\oplus\mathfrak{g}_{-1}\oplus\mathfrak{g}_0$ *for any* $i\geq 1$.

The Lie algebra $\mathfrak{G}_{k,l}$ is finite dimensional for any (k,l). We will describe it explicitly in the next subsection. Moreover, to any pair (k,l) one can assign in a natural way a $(3,6,\ldots)$-distribution of maximal class and type (k,l) such that its algebra of infinitesimal symmetries is equal to $\mathfrak{G}_{k,l}$. For this let $\mathfrak{S}_{k,l}^0$ be a subgroup of $\mathfrak{S}_{k,l}$, preserving the filtration (1.1), i.e., one point of the flat curve, and $\mathfrak{s}_{k,l}^0$ be the corresponding Lie algebras. Consider the following subalgebra $\mathfrak{p}_{k,l}$ of $\mathfrak{G}_{k,l}$

$$\mathfrak{p}_{k,l}=\mathbb{R}\eta\oplus V^{(-1)}\oplus\mathfrak{s}_{k,l}^0\oplus\bigoplus_{i\geq 1}\mathfrak{g}_i,$$

where $V^{(-1)}$ is as in the filtration (1.1).

Let $G_{k,l}$ be a Lie group with the Lie algebra $\mathfrak{g}$ and let $P_{k,l}$ be a subgroup corresponding to the subalgebra $\mathfrak{p}_{k,l}$. Then there is an invariant 3-dimensional distribution on $G_{k,l}/P_{k,l}$ that corresponds to the $P_{k,l}$-invariant subspace

$$\mathbb{R}\eta\oplus V^{(0)}\oplus\mathfrak{s}_{k,l}\oplus\bigoplus_{i\geq 1}\mathfrak{g}_i$$

in $\mathfrak{g}_{k,l}/\mathfrak{p}_{k,l}$ (note that $\dim\mathfrak{s}_{k,l}-\dim\mathfrak{s}_{k,l}^0=1$ and $\dim V^{(0)}-\dim V^{(-1)}=2$). We call this distribution *a flat* $(3,6,\ldots)$*-distributions of type (k,l).*

The flat $(3,6,\ldots)$-distribution of type (k,l) can be described in more explicit way as follows. Consider a Lie algebra $\mathfrak{m}_{k,l}$, generated as a vector space by elements X, $Y_i (1\leq i\leq k)$, $Z_j (1\leq j\leq k+l)$, η, with the following non-trivial Lie brackets:

$$\begin{aligned}
[X,Y_i]&=Y_{i+1},\quad i=1,\ldots,k-1;\\
[X,Z_j]&=Z_{j+1},\quad j=1,\ldots,k+l-1;\\
[Y_1,Z_1]&=\eta.
\end{aligned}$$

The flat $(3,6,\ldots)$-distributions of type (k,l) is equivalent to a left-invariant distribution on the Lie group $M_{k,l}$ with the Lie algebra $\mathfrak{m}_{k,l}$, which corresponds to the subspace $D_o=\langle X,Y_1,Z_1\rangle$.

The main result of the paper can be formulated as follows:

Main Theorem. *For any* $(3,6,\ldots)$*-distribution* D *of maximal class and type* (k,l) *there exists a natural fiber bundle* Q *over the manifold* N *equipped with a canonical frame. The dimension of* Q *is equal to the dimension of the Lie algebra* $\mathfrak{G}_{k,l}$. *There exists* $(3,6,\ldots)$*-distribution of maximal class and type* (k,l) *such that its algebra of infinitesimal symmetries is equal to* $\mathfrak{G}_{k,l}$

As a matter of fact if a $(3,6,\ldots)$-distribution D of maximal class and type (k,l) has the algebra of infinitesimal symmetries of dimension equal to $\dim\mathfrak{G}_{k,l}$, then D is locally equivalent to the flat $(3,6,\ldots)$-distribution of maximal class and type (k,l). This uniqueness statement will be proved in the forthcoming paper.

1.5. **Description of algebras $\mathfrak{s}_{k,l}$ and $\mathfrak{G}_{k,l}$.** To complete the picture we describe explicitely the symbols of our frame bundles (isomorphic to algebra of infinitesimal symmetries $\mathfrak{s}_{k,l}$ of the flat curve of of flags of type (k,l)) and the corresponding universal prolongation algebra $\mathfrak{G}_{k,l}$ from the main theorem. The cases of rectangular and non-rectangular diagrams are essentially different and considered separately

(a) The case of rectangular diagram. In this case $l = 0$. Let $V_{2k-1} = \langle E_1, \dots, E_{2k-1}\rangle$ be the irreducible $(2k-1)$-dimensional $\mathfrak{sl}(2,\mathbb{R})$-module with weight spaces $\langle E_i\rangle$ and $\mathbb{R}^2 = \langle \varepsilon_1, \varepsilon_2\rangle$ be the standard $\mathfrak{gl}(2,\mathbb{R})$-module. Identify the space V with the $\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{gl}(2,\mathbb{R})$ module $V_{2k-1}\otimes\mathbb{R}^2$, such that $e_i = E_i\otimes\varepsilon_1$ and $f_i = E_i\otimes\varepsilon_2$. Then the algebra $\mathfrak{s}_{k,0}$ is equal to the image in $\mathfrak{gl}(V)$ of the representation of the algebra $\mathfrak{sl}(2,r)\oplus\mathfrak{gl}(2,r)$, corresponding to the module $V_{2k-1}\otimes\mathbb{R}^2$.

Further, it turns out that $\mathfrak{G}_{2,0} = \mathfrak{so}(4,3)$ (compare with [1]). On the other hand, for $k \geq 3$, $l = 0$ the first prolongation $\mathfrak{g}_1$ of the algebra (1.4) is equal to 0 so that

$$\mathfrak{G}_{k,0} = \mathbb{R}\eta \oplus V \oplus \mathfrak{s}_{k,0}, \quad k \geq 3. \tag{1.5}$$

(b) The case of non-rectangular diagram. The structure of the Lie algebras $\mathfrak{s}_{k,l}$ and $\mathfrak{G}_{k,l}$ in case $l > 0$ is much more complicated and can be defined via the language of symplectic differential geometry. Let $r = 2k + l - 1$. Fix a (formal) coordinate system $(x_1, \dots, x_r, p_1, \dots, p_r)$ in the symplectic space of dimension $\mathbb{R}^{2r}$ with the symplectic form:

$$dx_1 \wedge dp_r - dx_1 \wedge dp_{r-1} + \cdots + (-1)^{r+1} dx_r \wedge dp_1.$$

Introduce the Poisson Lie bracket on the algebra of polynomials $\mathbb{R}[x_i, p_j]$. We shall define $\mathfrak{G}_{k,l}$ as a one-dimensional extension of a certain subalgebra in this algebra.

Let $\mathbb{P}^{r-1}$ be the projective space with homogeneous coordinates $[x_1 : x_2 : \ldots : x_r]$. Denote by C the normal rational curve in $\mathbb{P}^{r-1}$ given as an image of the Veronese embedding

$$\mathbb{P}^1 \to \mathbb{P}^{r-1}, \quad [s:t] \mapsto [s^{r-1} : s^{r-2}t : \ldots : t^{r-1}].$$

There is a unique irreducible $SL(2,R)$-action on $\mathbb{R}^r$ such that this rational normal curve is exactly an orbit of the highest vector in $\mathbb{R}^r$ under this action.

Denote by $\mathcal{T}^bC$ the b-th tangential developable variety of C. Here we assume that $\mathcal{T}^0C = C$. If $\mathcal{V}$ is any algebraic variety in $\mathbb{P}^{r-1}$, we denote, as usual, by $I(\mathcal{V})$ the ideal of homogeneous polynomials in $x_1, \dots, x_r$ vanishing on $\mathcal{V}$. We shall also denote by $I_b(\mathcal{V})$ the subspace of all polynomials of degree b in $I(\mathcal{V})$. Denote also by $\mathcal{S}_b\mathcal{V}$ the b-th secant variety of $\mathcal{V}$, which is defined as an algebraic closure of the union of $(n-1)$-planes in $\mathbb{P}^{r-1}$ passing through b points from $\mathcal{V}$. By definition we set $\mathcal{S}_1(\mathcal{V}) = \mathcal{V}$ and $\mathcal{S}_i(\mathcal{V}) = \emptyset$ for $i \leq 0$.

Now define a subalgebra $\tilde{\mathfrak{g}}$ in $\mathbb{R}[x_i, p_j]$ in $\mathbb{R}[x_i, p_j]$ as a sum of the following several subspaces in $\mathbb{R}[x_i, p_j]$:

- the subalgebra $\mathfrak{gl}(2,\mathbb{R})$ spanned by:

$$\begin{aligned}
X &= (r-1)x_2p_r - (r-2)x_3p_{r-1} + \cdots + (-1)^{r-1}x_rp_2;\\
Y &= x_1p_{r-1} - 2x_1p_{r-2} + \cdots + (-1)^{r-1}(r-1)x_{r-1}p_1;\\
H &= (r-1)x_1p_r - (r-3)x_2p_{r-1} + \cdots + (-1)^{r-1}(1-r)x_rp_1;\\
Z &= x_1p_r - x_2p_{r-1} + \ldots (-1)^{r-1}x_rp_1.
\end{aligned}$$

- $\langle p_1, \dots, p_n\rangle$;

- $I_s = I_s(\mathcal{V}_s) \subset \mathbb{R}[x_1, \dots, x_r]$, where $\mathcal{V}_s = \mathcal{S}_{s-1}(\mathcal{T}^{k-2}C)$ is the $(s-1)$-th secant variety of $(k-2)$-th tangential variety to the rational normal curve C. In particular, we have:

$$I_0 = \langle 1 \rangle;$$
$$I_1 = \langle x_1, \dots, x_r \rangle;$$
$$I_2 = \text{quadratic polynomials vanishing at } \mathcal{T}^{k-2}C;$$
$$I_3 = \text{cubic polynomials vanishing at } \mathcal{S}_2(\mathcal{T}^{k-2}C);$$
$$\dots$$

Note that in the extreme case of $k = 2$ we have $\mathcal{T}^{k-2}C = C$, and, for example, I_2 is spanned by all quadratic polynomials of the form $x_i x_j - x_k x_l$, $i + j = k + l$. In general, for any $k \geq 2$ the tangential variety $\mathcal{T}^{k-2}C$ does not lie in any proper linear subspace of $\mathbb{P}^{r-1}$. Therefore $\mathcal{S}_{s-1}(\mathcal{T}^{k-2}C) = \mathbb{P}^{r-1}$ for sufficiently large s, and $I_s = 0$. Thus, $\tilde{\mathfrak{g}}$ is always finite-dimensional, though its dimension depends exponentially on r and therefore on the dimension of the original manifold M (for a fixed $k \geq 2$).

The algebra $\mathfrak{s}_{k,l}$ is equal to $\langle X, Y, H, Z, \mathrm{Id} \rangle \oplus I_2$. The algebra $\mathfrak{G}_{k,l}$ is isomorphic to a one-dimensional extension of the Lie algebra $\tilde{\mathfrak{g}}$ by a semisimple element Z' such that

$$[Z', f] = (\deg(f) - 2) f$$

for any element $f \in \tilde{\mathfrak{g}} \subset \mathbb{R}[x_i, p_j]$.

Finally note that the main result of [3] and [4] about rank 2 distributions on an n dimensional manifold can be formulated in a similar way as in our Main Theorem here. The Young diagram of the linearizations consists of one row, the corresponding flat curve is a curve of complete flags, consisting of all osculating subspaces of the rational normal curve in the projective space $\mathbb{P}^{2n-7}$, the algebra of its infinitesimal symmetries is equal to the image of the irreducible embedding of $\mathfrak{gl}(2, \mathbb{R})$ into $\mathfrak{s}_n = \mathfrak{gl}(V)$, where $\dim V = 2n - 6$, the flat distribution and its algebra of symmetries is described as in the subsection 1.4 (d), replacing $\mathfrak{s}_{k,l}$ by $\mathfrak{s}_n$. In particular, the algebra of infinitesimal symmetries is equal to the Tanaka universal prolongation of $\mathbb{R}\eta \oplus V \oplus \mathfrak{s}_n$, which is equal to $\mathbb{R}\eta \oplus V \oplus \mathfrak{s}_n$ itself for $n > 5$ and to the exceptional simple Lie algebra G_2 for $n = 5$. All this suggests that our Main Theorem can be generalized to much general situation of distributions of arbitrary rank.

2. Symplectification procedure

2.1. **Description of a characteristic line distribution.** Let us describe the characteristic line distribution $\mathcal{C}$ from subsection 1.1 in terms of a local basis (X_1, X_2, X_3) of the distribution D, $D(q) = \mathrm{span}\{X_1(q), X_2(q), X_3(q)\}$. Denote $X_{i,j} = [X_i, X_j]$ for $1 \leq i, j \leq 3$. Let us introduce the "quasi-impulses" $u_i : T^*M \mapsto \mathbb{R}$, $u_{i,j} : T^*M \mapsto \mathbb{R}$, $1 \leq i, j \leq 3$:

$$u_i(\lambda) = p \cdot X_i(q), \quad u_{i,j}(\lambda) = p \cdot X_{i,j}(q) \quad \lambda = (p, q),\ q \in M,\ p \in T_q^*M. \tag{2.1}$$

Then by definitions

$$D^{\perp} = \{\lambda \in T^*M : u_1(\lambda) = u_2(\lambda) = u_3(\lambda) = 0\}. \tag{2.2}$$

As usual, for given function $G : T^*M \mapsto \mathbb{R}$ denote by $\vec{G}$ the corresponding Hamiltonian vector field defined by the relation $i_{\vec{G}}\hat{\sigma} = -dG$.

Lemma 1. *The characteristic line distribution $\hat{\mathcal{C}}$ on $D^{\perp} \backslash (D^2)^{\perp}$ satisfies*

$$\hat{\mathcal{C}} = \langle u_{2,3}\vec{u}_1 - u_{1,3}\vec{u}_2 + u_{1,2}\vec{u}_3 \rangle. \tag{2.3}$$

Proof. Take a vector field H on $D^\perp\backslash(D^2)^\perp$ such that locally $\widehat{\mathcal{C}}(\lambda) = \{\mathbb{R}H(\lambda)\}$. Then by definition of $\widehat{\mathcal{C}}$ we have $i_H\hat\sigma|_{(D^2)^\perp} = 0$. From this and (2.2) it follows that $i_H\hat\sigma \in \langle du_1, du_2, du_3\rangle$, which implies that

$$H \in \langle \vec{u}_1, \vec{u}_2, \vec{u}_3\rangle. \tag{2.4}$$

On the other hand, H is tangent to $D^\perp$, i.e $du_j(H) = 0$ for $1 \le j \le 3$. This and (2.4) easily implies (2.3). □

Let $D'(q) \subset D(q)$ be the union of one-dimensional subspaces $\pi_*(\mathcal{C}(\lambda))$ is equal to $D(q)$:

$$D'(q) = \left\{\pi_*(\widehat{\mathcal{C}}(\lambda)) \,:\, \lambda \in D^\perp\backslash(D^2)^\perp, \pi(\lambda) = q\right\}. \tag{2.5}$$

As a consequence of the previous if $\dim D^2(q) = 6$, then

$$D'(q) = D(q) \tag{2.6}$$

In particular, in this case the original distribution can be recovered from its characteristic line distribution.

Similarly, from (2.3) it follows that if $\dim D^2(q) = 4$ or $\dim D^2(q) = 5$, then the set $D'(q)$ constitutes a one-dimensional or a two-dimensional subspace of $D(q)$ respectively. If $\dim D^2(q) = 4$ then a line distribution D' is a characteristic subdistribution of D, i.e. $[D', D] \subset D$. In this case we can make, at least locally, a factorization of M by the $1-$foliation generated by the line distribution $D'(q)$ so that in the quotient manifold we get the rank 2 distribution $D(q)/D'(q)$. In this way we reduce the equivalence problem for the original rank 3 distribution D to the equivalence problem for certain rank 2 distribution, which was treated in [3, 4] If $\dim D^2(q) = 5$, then it can be shown that the rank 2 subdistribution D' satisfies $(D')^2 \subsetneq D$ and this is the unique rank 2 subdistribution, satisfying this property. Therefore, sometimes it is called the *square root* of the distribution D. There are two possibilities: either $(D')^2(q) = D(q)$ for generic q on M or $D'(q)$ is involutive, i.e. $(D')^2 = D'$. In the first case the equivalence problem for rank 3 distributions is reduced to the equivalence problem for rank 2 distributions, which was treated in [3, 4].

So, the equivalence problem for rank 3 distributions cannot be reduced to one for rank 2 distributions only in the following two cases: $\dim D^2 = 6$ or $\dim D^2 = 5$ and the square root D' is involutive. In the present paper we restrict ourselves to the case $\dim D^2(q) = 6$, i.e. to $(3, 6, \dots)-$distributions

2.2. **The curves of flags associated with abnormal extremals.** In what follows the canonical projection from $\mathbb{P}D^\perp$ to M will be denoted also by π. By analogy with [3] and [4], let $\mathcal{J}$ be the pull-back of the distribution D on $\mathbb{P}D^\perp\backslash\mathbb{P}(D^2)^\perp$ by the canonical projection π:

$$\mathcal{J}(\lambda) = \{v \in T_\lambda(\mathbb{P}D^\perp) : \pi_* v \in D(\pi(\lambda))\}. \tag{2.1}$$

Note that $\dim \mathcal{J} = n - 1$, $\mathcal{J} \subset \widetilde{\Delta}$, and $\mathcal{C} \subset \mathcal{J}$ by (2.3) . The distribution $\mathcal{J}$ is called the *lift of distribution D to $\mathbb{P}D^\perp\backslash\mathbb{P}(D^2)^\perp$*.

In the sequel we shall work with the lift $\mathcal{J}$ instead of the original distribution D. The crucial advantage of working with $\mathcal{J}$ is that it has the distinguished line sub-distribution $\mathcal{C}$, while the original distribution D has no distinguished sub-distributions in general.

We can produce a monotonic (by inclusion) sequence of distributions (in general of nonconstant ranks) by making iterative Lie brackets of $\mathcal{C}$ and $\mathcal{J}$. Namely, first define a sequence of subspaces $\mathcal{J}^{(i)}(\lambda)$, $\lambda \in \mathbb{P}D^\perp\backslash\mathbb{P}(D^2)^\perp$, by the following recursive formulas:

$$\mathcal{J}^{(0)} = \mathcal{J}, \quad \mathcal{J}^{(i)} = \mathcal{J}^{(i-1)} + [\mathcal{C}, \mathcal{J}^{(i-1)}], \quad \forall\, i \ge 1. \tag{2.2}$$

$$\mathcal{J}^{(-i)}(\lambda) = \left\{ v \in \mathcal{J}^{(1-i)}(\lambda) : \begin{array}{l} \exists \text{ a vector field } \mathcal{V} \subset \mathcal{J}^{(1-i)} \text{ with } \mathcal{V}(\lambda) = v \\ \text{such that } \big[\mathcal{C}, \mathcal{V}\big](\lambda) \in \mathcal{J}^{(1-i)}(\lambda), \end{array} \right\} \quad \forall i \geq 1. \tag{2.3}$$

It is easy to show that in (2.3) one can replace the quantor $\exists$ by $\forall$. It is clear by constructions that $\mathcal{J}^{(i)}(\lambda) \subseteq \mathcal{J}^{(i+1)}(\lambda)$ for all $i \in \mathbb{Z}$. Besides, $\mathcal{J}^{(i)} \subset \widetilde{\Delta}$ for any $i \in \mathbb{Z}$, because $\mathcal{J} \subset \widetilde{\Delta}$ and $\mathcal{C}$ is the characteristic subdistribution of $\widetilde{\Delta}$. Thus we get a flag

$$\ldots \subseteq \mathcal{J}^{(-i)}(\lambda) \subseteq \ldots \subseteq \mathcal{J}^{(-1)}(\lambda) \subset \mathcal{J}^{(0)}(\lambda) \subset \mathcal{J}^{(1)}(\lambda) \subseteq \ldots \subseteq \mathcal{J}^{(i)}(\lambda) \subseteq \ldots \subset \widetilde{\Delta}(\lambda) \tag{2.4}$$

Further, by analogy with [12] and [13], the following identity holds

$$\mathcal{J}^{(-1-i)}(\lambda) = \{v \in \widetilde{\Delta}(\lambda) : \tilde{\sigma}(v, w) = 0 \ \forall w \in \mathcal{J}^{(i)}(\lambda)\}, \quad \forall\, i \geq 0, \tag{2.5}$$

where as before $\tilde{\sigma}$ is the antisymmetric form defined on each subspace of the distribution $\widetilde{\Delta}$ canonically up to a multiplication by a constant.

We summarize the main properties of the sequences $\{\mathcal{J}^{(i)}\}_{i \in Z}$ in the following:

Proposition 1.

(1) $\dim \mathcal{J}^{(1)}(\lambda) - \dim \mathcal{J}(\lambda) \leq 2$ *and the equality holds iff* $\dim D^2\big(\pi(\lambda)\big) = 6$*,*
(2) $\dim \mathcal{J}^{(i)}(\lambda) - \dim \mathcal{J}^{(i-1)}(\lambda) \leq 2$*, for any* $i \in Z$
(3) $[\mathcal{C}, \mathcal{J}^{(i-1)}] \subseteq \mathcal{J}^{(i)}$ *and* $[\mathcal{C}, \mathcal{J}^{(i-1)}] = \mathcal{J}^{i}$ *if and only if either* $i \geq 1$ *or* $\dim \mathcal{J}^{(i)} - \dim \mathcal{J}^{(i-1)} = \dim \mathcal{J}^{(i-1)} - \dim \mathcal{J}^{(i-2)}$ *for* $i \leq 0$*.*

Proof. Given a point $\lambda \in \mathbb{P}D^{\perp} \backslash \mathbb{P}(D^2)^{\perp}$ take a vector field H on M tangent to D such that $\pi_* C(\lambda) = \{\mathbb{R}H\big(\pi(\lambda)\big)\}$. Then directly from definition it follows that

$$\mathcal{J}^{(1)}(\lambda) = \{v \in T_\lambda(\mathbb{P}D^{\perp}) : \pi_*\, v \in [H, D]\big(\pi(\lambda)\big)\}.$$

This implies property (1) of the proposition. Property (2) for $i \geq 2$ follows from the first relation of Property (1) and the fact that the distribution $\mathcal{C}$ has rank 1. Property (2) for $i \leq 0$ follows from relation (2.5). Property (3) follows easily from definitions (2.2) and (2.3). $\square$

The dynamics of the flags (2.4) along any abnormal extremal defines certain curve of flags of isotropic and coisotropic subspaces in a linear symplectic space. More precisely, let γ be a segment of abnormal extremal of D and O_γ be a neighborhood of γ in $\mathbb{P}D^{\perp}$ such that the factor $N = O_\gamma/(\textit{the characteristic one-foliation})$ is a well defined smooth manifold. Let $\Phi : O_\gamma \to N$ be the canonical projection on the factor. For each $i \in \mathbb{Z}$ we can define the following curves of subspaces in $T_\gamma N$:

$$\lambda \mapsto J^{(i)}(\lambda) \stackrel{def}{=} \Phi_*\big(\mathcal{J}^{(i)}(\lambda)\big), \tag{2.6}$$

Note also that $\Delta \stackrel{def}{=} \Phi_* \widetilde{\Delta}$ is well defined contact distribution on N and the form $\tilde{\sigma}$ induces on each space $\Delta(\gamma)$ the canonical, up to a multiplication by a constant, symplectic form, which will be denoted by σ. From (2.4) it follows that $J^{(i)}(\lambda) \subset \Delta(\gamma)$ for all $i \in \mathbb{Z}$ and $\lambda \in \gamma$. By constructions,

$$J^{(-1-i)}(\lambda) = \big(J^{(i)}(\lambda)\big)^{\angle} \tag{2.7}$$

Moreover, the subspaces $J^{(i)}(\lambda)$ are coisotropic w.r.t. the form σ for $i \geq 0$ and isotropic for $i < 0$. Besides, $\dim\, \Delta(\gamma) = 2(n-3)$ and $\dim J^{(0)}(\lambda) = n-2$.

The curve

$$\lambda \mapsto \Big\{ \ldots \subseteq J^{(-i)}(\lambda) \subseteq \ldots \subseteq J^{(-1)}(\lambda) \subset J(\lambda) \subset J^{(1)}(\lambda) \subseteq \ldots \subseteq J^{(i)}(\lambda) \subseteq \ldots \Big\}, \quad \lambda \in \gamma, \tag{2.8}$$

of flags of isotropic and coisotropic subspaces in a linear space $\Delta(\gamma)$ will be called *the linearization* of the flag $\{\mathcal{J}^{(i)}\}$ along the characteristic curve γ. Clearly, any invariant of such curve w.r.t. the action of the group $\mathrm{CSp}\big(\Delta(\gamma)\big)$ of linear transformations of $\Delta(\gamma)$, preserving the form σ up to a multiplication by a constant, automatically produces an invariant of the distribution D itself. Moreover, it turns out that under certain generic assumptions one can construct the canonical frames of the distribution D from the study of bundles of moving frames canonically associated with such curves.

The point $\lambda \in \mathbb{P}D^\perp \backslash \mathbb{P}(D^2)^\perp$ is called regular if there is $i(\lambda) \in \mathbb{N}$ such that $J^{i(\lambda)}(\lambda) = \Delta(\gamma)$, where γ is the abnormal extremal, containing λ. The point λ is called strongly regular, if it is regular and there is a neighborhood U of λ in $\mathbb{P}D^\perp$ such that for any $i \in \mathbb{N}$ dimensions of $J^{(i)}(\bar\lambda)$ does not depend on $\bar\lambda \in \gamma \cap U$. Strong regularity implies that

$$\dim J^{(i+1)}(\lambda) - \dim J^{(i)}(\lambda) \leq \dim J^{(i)}(\lambda) - \dim J^{(i-1)}(\lambda). \tag{2.9}$$

and we define as follows the *Young diagram T of the curve $J^{(0)}$ at λ*: for $i \geq 1$ the number of boxes in the ith column of T is equal to $\dim J^{(i)}(\lambda) - \dim J^{(i-1)}(\lambda)$. A strongly regular point λ such that the Young diagram of the curve $J^{(0)}$ at λ is equal to T will be also called *T-regular*.

Rewriting Properties (1)-(2) of Proposition 1 in terms of subspaces $J^{(i)}(\lambda)$, we get

Proposition 2.

(1) $\dim J^{(1)}(\lambda) - \dim J(\lambda) \leq 2$ *and the equality holds iff* $\dim\, D^2\big(\pi(\lambda)\big) = 6$*,*
(2) $\dim J^{(i)}(\lambda) - \dim J^{(i-1)}(\lambda) \leq 2$*, for any* $i \in Z$

The last proposition means that columns of the Young diagrams of linearizations along abnormal extremals of rank 3 distributions have either one or two boxes and the first column has two boxes if and only if $\dim\, D^2\big(\pi(\lambda)\big) = 6$.

Definition 1. *A rank 3 distribution D is said to be of maximal class at a point $q \in M$, if there exists at least one strongly regular point $\lambda \in \mathbb{P}D^\perp \backslash \mathbb{P}(D^2)^\perp$ such that $\pi(\lambda) = q$.*

Since distributions $\mathcal{J}$ and $\mathcal{C}$ are algebraic on each fiber of $\mathbb{P}D^\perp$, the maximality of class at a point q implies that there is a Young diagram T such that the set of T-regular points of the fiber $\pi^{-1}(q) \cap \mathbb{P}D^\perp$ is a nonempty Zariski open subset. In this case T will be called the *diagram of the distribution D at the point q*.

In the sequel we will study rank 3 distributions of maximal class with $\dim\, D^2 = 6$ and with the fixed diagram T, i.e. $(3, 6, \ldots)$ distributions. The number of columns in the diagram T, consisting of one box, is called the *shift* of the diagram T and it will be denoted by l. We also assume that the number of remaining columns (all of which consisting of 2 boxes) is equal to $k-1$. In this case we also say that the diagram T is of type (k, l). From the assumption that $\dim\, D^2 = 6$ it follows that $k \geq 2$. Note that the number of boxes in T is equal to $\mathrm{rank}\, \Delta(\gamma) - \mathrm{rank}\, J^{(0)}(\lambda) = n - 4$. Therefore $n = 2k + l + 2$ and the parity of l is equal to the parity of n (recall that n is the dimension of the ambient manifold M).

Note that germs of rank 3 distributions of the maximal class are generic. Indeed, from algebraicity of distributions $\mathcal{J}$ and $\mathcal{C}$ on each fiber of $\mathbb{P}D^\perp$ it follows that the distribution has maximal class at a point q_0 if and only if its jet of sufficiently high order belongs to the Zariski open set of the jet space of this order. Therefore in order to prove the statement it is sufficient to give just one example of a germ of rank 3-distributions of the maximal class. The flat $(3, 6, \dots)$-distribution of type (k, l) with $2k + l + 2 = n$, define in the Introduction, provides such example.

Finally let us reformulate property (3) of Proposition 1 in terms of subspaces $J^{(i)}(\lambda)$. For this let us introduce some notation. Let $G_k(W)$ be the Grassmannian $G_k(W)$ of k-dimensional subspaces of a linear space W. Take a smooth (unparametrized) curve $\Lambda : \gamma \mapsto G_k(W)$. Let $\mathfrak{S}(\Lambda)$ be the set of all smooth curves $\ell : \gamma \mapsto W$ such that $\ell(\lambda) \in \Lambda(\lambda)$ for all $\lambda \in \gamma$. If $\varphi : \gamma \mapsto \mathbb{R}$ is a parametrization of γ, $\varphi(\lambda) = 0$ and $\psi = \varphi^{-1}$, then denote

$$\mathcal{D}^{(i)}\Lambda(\lambda) = \operatorname{span}\Big\{\frac{d^j}{dt^j}\ell\big(\psi(t)\big)|_{t=0} : \ell \in \mathfrak{S}(\Lambda),\, 0 \le j \le i\Big\}. \tag{2.10}$$

In particular, directly from definitions it follows that

$$J^{(i)}(\lambda) = \mathcal{D}^{(i)}J(\lambda). \tag{2.11}$$

Besides, as a consequence of property (3) of Proposition 1 we have the following

Proposition 3. *Assume that λ is strongly regular, then $\mathcal{D}^{(1)}J^{(i-1)}(\lambda) \subseteq J^{(i)}(\lambda)$ and $\mathcal{D}^{(1)}J^{(i-1)}(\lambda) = J^{(i)}(\lambda)$ if and only if either $i \ge 1$ or $\dim J^{(i)}(\lambda) - \dim J^{(i-1)}(\lambda) = \dim J^{(i-1)}(\lambda) - \dim J^{(i-2)}(\lambda)$ for $i \le 0$.*

3. Canonical bundles of moving frames for curves of flags associated with abnormal extremals

In the present section given the Young diagram T of type (k, l) with $k \ge 2$ we construct the canonical bundle of moving frames for any curve of flags (2.8) with the diagram T in a linear space $\Delta(\gamma)$. This gives automatically the canonical frame bundle for any rank 3 distribution with the diagram T on the contact distribution Δ of the manifold N constructed in the previous section. The main point is that this frame bundle has the constant symbol in a sense defined in the Introduction. This symbol is actually equal to the algebra of infinitesimal symmetries of the flat curve of flags, corresponding to the diagram T.

We will work with $J^{(i)}$ considered as a vector bundle over γ with the fiber $J^{(i)}(\lambda)$ over a point λ. The cases of rectangular and non-rectangular Young diagrams are quite different and will be considered separately.

3.1. **Curves of flags with rectangular diagram.** Assume that the shift l of the diagram T is equal to zero, i.e. T is a rectangle with 2 rows and $2k - 1$ columns. In this case $J^{(-k)}(\lambda) = 0$ and $\dim J^{(-k+1)}(\lambda) = 2$. By Proposition 3 we have $(J^{(-k+1)})^{(2k-2)}(\lambda) = \Delta(\gamma)$. First fix a parametrization $\varphi : \gamma \mapsto \mathbb{R}$ of γ. In the sequel, if s is a section of the bundle $J^{(i)}$, then the vector $s(t)$ belongs to $J^{(i)}\big(\varphi^{-1}(t)\big)$. The bundle $J^{(-k+1)}$ has a unique (w.r.t. the parametrization φ) connection such that any its horizontal section e satisfies

$$e^{(2k-1)}(t) \in J^{(2k-3)}\big(\varphi^{-1}(t)\big) \quad \forall\, t. \tag{3.12}$$

A tuple

$$\big(e_1(t), e_2(t), e_1'(t), e_2'(t), \dots, e_1^{(2k-2)}(t), e_2^{(2k-2)}(t)\big), \tag{3.13}$$

where e_1 and e_2 are horizontal sections of the canonical (w.r.t φ) connection on the bundle $J^{(-k+1)}$ is called the *normal (w.r.t φ) moving frame of the curve* (2.8) *generated by the pair* (e_1, e_2). All such frames (for the fixed parametrization φ) constitute the principle bundle over γ

with a structure group $GL(2,\mathbb{R})$. If e_1 and e_2 are two nonproportional horizontal sections of the canonical connection on the bundle $J^{(-k+1)}$, then

$$\begin{pmatrix} e_1^{(2k-1)}(t) \\ e_2^{(2k-1)}(t) \end{pmatrix} = \sum_{i=1}^{2k-3} A_i(t) \begin{pmatrix} e_1^{(i)}(t) \\ e_2^{(i)}(t) \end{pmatrix} \tag{3.14}$$

for some 2×2-matrices $A_i(t)$. Note that the operator $\mathcal{A}_i^{\varphi}(\lambda) : J^{(-k+1)}(\lambda) \mapsto J^{(-k+1)}(\lambda)$, having the matrix $A_i(t)$) w.r.t. the basis $\big(e_1(t), e_2(t)\big)$, where $t = \varphi(\lambda)$ does not depend on the choice of the sections e_1 and e_2 as above. The operators $\mathcal{A}_i^{\varphi}(\lambda)$ are invariants of the parameterized curve $t \mapsto J^{(-k+1)}\big(\varphi^{-1}(t)\big)$.

Now we will show that the curve (2.8) (or equivalently the curve γ) can be endowed with the canonical projective structure, i.e., we have a distinguished set of parameterizations (called projective) such that the transition function from one such parameterization to another is a Möbius transformation. For this let

$$\rho_{1,\varphi}(\lambda) = \operatorname{tr}\big(\mathcal{A}_{2k-3}^{\varphi}(\lambda)\big). \tag{3.15}$$

How $\rho_{1,\varphi}$ transforms under reparametrization of γ? Let φ_1 be another parametrization and $v = \varphi \circ \varphi_1^{-1}$. Then it is not hard to show that $\rho_{1,\varphi}$ and ρ_{1,φ_1} are related as follows:

$$\rho_{1,\varphi_1}(\lambda) = v'(\tau)^2 \rho_{1,\varphi}(\lambda) + C_k^1 \mathbb{S}(v)(\tau), \quad \tau = \varphi_1(\lambda) \tag{3.16}$$

where $\mathbb{S}(v)$ is a Schwarzian derivative of v, i.e. $\mathbb{S}(v) = \frac{d}{dt}\left(\frac{v''}{2\,v'}\right) - \left(\frac{v''}{2\,v'}\right)^2$ and C_k^1 is a nonzero constant. From the last formula and the fact that $\mathbb{S}v \equiv 0$ if and only if the function v is Möbius it follows that *the set of all parameterizations φ of γ such that*

$$\rho_{1,\varphi} \equiv 0 \tag{3.17}$$

defines the canonical projective structure on γ. Such parameterizations are called the *projective parameterizations of the abnormal extremal γ*. The set of the normal moving frames of the curve (2.8) w.r.t. all projective parameterizations constitute the principle bundle over γ with a structure group $\mathrm{ST}(2,\mathbb{R})\times\mathrm{GL}(2,\mathbb{R})$, where $\mathrm{ST}(2,\mathbb{R})$ denotes the group of lower triangular real 2×2 matrices with unit determinant.

Now let, as before, N be a manifold of all leaves of the characteristic foliation $\mathcal{C}$ in a small neighborhood of a point $\lambda_0 \in \mathbb{P}D^{\perp}\backslash\mathbb{P}(D^2)^{\perp}$ such that the linearizations of the flag $\{\mathcal{J}^{(i)}\}$ along any leaf of $\mathcal{C}$ in this neighborhood has the Young diagram T of type $(k,0)$. Consider the fiber bundle $P_{k,0}$ over N such that its fiber over a point γ is the set of tuples $(\gamma, \varphi, (e_1, e_2))$, where φ is a projective parametrization on the leaf (the abnormal extremal) γ and (e_1, e_2) is a basis of $J^{(-k+1)}\big(\varphi^{-1}(0)\big)$. By construction, $P_{k,0}$ is a principle bundle with a structure group $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{GL}(2,\mathbb{R})$.

To any point $\Gamma \in P_{k,0}$, $\Gamma = \big(\gamma, \varphi, (e_1, e_2)\big)$, assign the frame $\mathfrak{F}_1(\Gamma)$ on the space $\Delta(\gamma)$ which is equal to the value at 0 of normal w.r.t φ moving frame of the curve (2.8) generated by the pair of horizontal sections equal to (e_1, e_2) at 0. Actually, $\mathfrak{F}_1$ maps the fiber of $P_{k,0}$ over γ to the space of all frames on $\Delta(\gamma)$. It is easy to see that this mapping is an injective immersion. So, the fiber of $P_{k,0}$ over γ can be identified with its image under $\mathfrak{F}_1$ and we can look on $P_{k,0}$ as on a bundle of frames on the contact distribution Δ on N.

3.2. **Curves of flags with non-rectangular diagram.**

In this case $\dim J^{(i)}(\lambda) = \dim J^{(i)}(\lambda) + 1$ for any $-k-l+1 \le i \le -k$, while $J^{(-k-l)} = 0$ and $\dim J^{(-k+1)}(\lambda) = \dim J^{(-k)}(\lambda) + 2$. Take a nonzero section e of $J^{(-k-l+1)}$ and a nonzero section f of $J^{(-k+1)}$ such that $J^{(-k+1)}(\lambda) = \big(J^{(-k)}\big)^{(1)}(\lambda) \oplus \{\mathbb{R}f(\lambda)\}$. A pair (e, f) is said to be a *principal pair of sections of the curve* (2.8). As before, fix a parametrization $\varphi : \gamma \mapsto \mathbb{R}$ of γ. By Proposition 3 the whole curve of flags (2.8) can be recovered from the sections e and f by differentiation and the tuples

$$\big(e(t), e'(t), \dots, e^{(2k+l-2)}(t), f(t), f'(t), \dots, f^{(2k+l-2)}(t)\big) \tag{3.18}$$

constitute a moving frame in $\Delta(\gamma)$. This moving frame is said to be *corresponding to the pair* (e, f) *and parametrization* φ. Fix one symplectic form σ from the one-parametric family of symplectic forms on $\Delta(\gamma)$.

We start with the following

Definition 2. *A frame* $\Big(e_1, \dots, e_{2k+l-1}, f_1, \dots, f_{2k+l-1}\Big)$ *of the symplectic space* $(\Delta_\gamma, \widetilde{\sigma})$ *is said to be* (k, l)*-quasisymplectic if the following conditions hold:*

1. $\sigma(e_i, e_j) = 0$ *for all* $i + j \le 2k + 2l$, $\sigma(e_i, f_j) = 0$ *for all* $i + j \le 2k + l - 1$, $\sigma(f_i, f_j) = 0$ *for all* $i + j \le 2k$;
2. $\sigma(f_i, e_{2k+l-i}) = (-1)^{i-k}$ *for all* $1 \le i \le 2k + l - 1$;
3. $\sigma(f_i, e_{2k+l+1-i}) = 0$ *for all* $2 \le i \le 2k + l - 1$;
4. $\sigma(f_{k+i}, f_{k+i+1}) = 0$ *for all* $0 \le i \le l$.

As before, first we will fix a parametrization $\varphi : \gamma \mapsto \mathbb{R}$ of γ. Note that if (e, f) and $(\widetilde{e}, \widetilde{f})$ are two principal pairs of sections of the curve (2.8), then they are related as follows:

$$\widetilde{e}(t) = \alpha_1(t)e(t), \quad \widetilde{f}(t) = \alpha(t)f(t) + \sum_{i=1}^{l+1} \beta_i(t)e^{(i-1)}(t) \tag{3.19}$$

for some functions α, α_1, $\beta_1, \beta_2, \dots, \beta_{l+1}$, where $\alpha(t) \ne 0$ and $\alpha_1(t) \ne 0$ for any t.

A principal pair (e, f) of sections of the curve (2.8) is called a *normal pair of sections associated with the parametrization* φ *and the symplectic form* σ, if the corresponding moving frame (3.18) is (k, l)-quasisymplectic for any t. In this case the moving frame (3.18) is said to be *normal moving frame of the curve* (2.8)*, generated by the normal pair of sections* (e, f) *and associated with the parametrization* φ *and the symplectic form* σ. We also say that a germ (e, f) of a pair of sections at a point $\lambda_0 \in \gamma$ and the corresponding germ of a moving frame (3.18) at λ is *normal* if their representatives are normal for a restriction of the curve (2.8) on a neighborhood of λ_0.

Theorem 1. *For any point* $\lambda_0 \in \gamma$ *the set of normal germs at* λ_0 *of a pair of sections associated with the parametrization* φ *and the symplectic form* σ *is not empty. If* (e, f) *is a germs at* λ_0 *of a pair of sections associated with the parametrization* φ *and the symplectic form* σ*, then given a nonzero real number* C *and a set of real numbers* $\{a_{r,i} : 0 \le r \le [\frac{l}{2}], 0 \le i \le 2l - 4r\}$ *there exists a unique normal germ of a pair of sections* $(\widetilde{e}, \widetilde{f})$ *associated with the parametrization* φ *and the symplectic form* σ *such that* (e, f) *and* $(\widetilde{e}, \widetilde{f})$ *are related by formulas* (3.19) *with*

$$\alpha(t_0) = C, \quad \beta^{(i)}_{l+1-2r}(t_0) = a_{i,r}, \quad \forall\, 0 \le r \le \left[\frac{l}{2}\right], 0 \le i \le 2l - 4r, \tag{3.20}$$

where $t_0 = \varphi(\lambda_0)$.

Proof. The moving frame (3.18), corresponding to a principal tuple (e, f) satisfies condition of Definition 2 automatically: the first condition follows from the fact that spaces $J^{(-i)}(\lambda)$ are isotropic for $i \ge 1$ and from (2.7). If a nonzero section e of $J^{(-k-l+1)}$ is fixed, then the condition $\sigma\big(f^{(k-1)}(t), e^{(k+l-1)}(t)\big) \equiv 1$ fixes $f(t)$ modulo $(J^{(-k)})^{(1)}(\varphi^{-1}(t))$ for a section f such that (e, f) is a principal pair associated with the parametrization φ and the symplectic form σ. From here and the fact that $J^{(-1-i)}(\lambda) = \big(J^{(i)}(\lambda)\big)^{\angle}$ it follows by differentiations that $\sigma\big(f^{(i-1)}(t), e^{(2k+l-i-1)}(t)\big) \equiv (-1)^{i-k}$, i.e. condition (2) of Definition 2 holds for the corresponding moving frame (3.18). Any two principal pairs (e, f) and $(\widetilde{e}, \widetilde{f})$ such that the corresponding moving frames satisfy condition (2) of Definition 2 are related as follows

$$\widetilde{e}(t) = \frac{1}{\alpha(t)} e(t), \quad \widetilde{f}(t) = \alpha(t) f(t) + \sum_{i=1}^{l+1} \beta_i(t) e^{(i)}(t) \tag{3.21}$$

for some functions α, $\beta_1, \beta_2, \dots, \beta_{l+1}$, where $\alpha(t) \neq 0$ for any t.

By direct computations one can easily get the following

Lemma 2. *The condition $\sigma\big(\widetilde{f}^{(k-1)}(t), \widetilde{e}^{(k+l)}(t)\big) \equiv 0$ is equivalent to the following relation:*

$$(2k+l-1)\alpha'(t) = \sigma\big(f^{(k-1)}(t), e^{(k+l)}(t)\big)\alpha(t) + \sigma\big(e^{(k+l-1)}(t), e^{(k+l)}(t)\big)\beta_{l+1}(t). \tag{3.22}$$

It is convenient to introduce the notion of weights for sections e, $\widetilde{e}$, f, $\widetilde{f}$ and their derivatives, functions α, β_i and their derivatives, symplectic products of sections, produced by multiplications of these functions on these sections, and products of all of the above. Namely, we set

$$\deg e^{(i)} = \deg \widetilde{e}^{(i)} = -k - l + i + 1, \quad \deg f^{(i)} = \deg \widetilde{f}^{(i)} = -k + i + 1 \tag{3.23}$$

Further, set

$$\deg \beta_i^{(s)} \stackrel{def}{=} l + 1 - i + s, \quad \deg \alpha^{(s)} \stackrel{def}{=} s. \tag{3.24}$$

Then deg of any object as above will be called the *weight or the degree* of this object. Finally, the weight of the product of objects above is by definition a sum of weights of all its factors and for any pair of sections v_1, v_2, obtained by multiplication of functions from (3.24) on sections from (3.23), we set $\deg \sigma(v_1, v_2) = \deg v_1 + \deg v_2$. The fact that $J^{(-1+i)} = (J^{(i)})^{\angle}$ can be written in terms of weights as follows: for any pair of sections v_1 and v_2 as above

$$\sigma(v_1, v_2) = 0, \quad \text{if } \deg v_1 + \deg v_2 < 0. \tag{3.25}$$

The reason for the definition of weights is that we want the righthand sides of relations in (3.21) and their derivatives to be homogeneous of the same degree as the lefthand sides and their corresponding derivatives.

Now assume that (e, f) and $(\widetilde{e}, \widetilde{f})$ are two principal pairs associated with parametrization φ and the symplectic form σ such that the corresponding moving frames satisfy Conditions (1) and (2) of Definition 2 and they are related by (3.21). Consider the following tuple of sections

$$S_i = \Big\{ f(t), \dots, f^{(i)}(t), e(t), e'(t), \dots, e^{(i+l)}(t) \Big\} \tag{3.26}$$

Lemma 3. *The condition $\widetilde{\sigma}(\widetilde{f}^{(k+j-1)}(t), \widetilde{f}^{(k+j)}(t)) = 0$ can be expressed as the following differential equation w.r.t. the functions β_i:*

$$\alpha(t) \sum_{i=1}^{l+1} \left(\binom{k+j-1}{2j+i-l} + \binom{k+j}{2j+i-l} \right) \beta_i^{(2j+i-l)}(t) = \Phi_j \tag{3.27}$$

where Φ_j is a polynomial expression w.r.t. the functions $\{\beta_i^{(s)}(t)\}_{1 \leq i \leq l+1, 0 \leq s < 2j+i-l}$, $\{\alpha^{(i)}(t)\}_{i=0}^{\min\{k+j, 2j+1\}}$, and symplectic products with positive weights of pairs of sections from $S_{k+j-1} \times S_{k+j}$. Moreover, *the monomials of* Φ_j *are quadratic w.r.t. the functions* $\{\beta_i^{(s)}(t)\}_{1\le i\le l+1, 0\le s<k+j}$, $\{\alpha^{(i)}(t)\}_{i=0}^{k+j}$ *and the weights of these monomials are equal to* $2j+1$.[1]

Proof. Replacing $\widetilde{f}^{(k+j)}(t)$ and $\widetilde{f}^{(k+j-1)}(t)$ in $\sigma(\widetilde{f}^{(k+j-1)}(t), \widetilde{f}^{(k+j)}(t))$, $j\ge 0$, by their expression in terms of the moving frame associated with the pair (e,f), functions β_i and α from (3.21), we get certain polynomial expression w.r.t. the functions $\{\beta_i^{(s)}(t)\}_{1\le i\le l+1, 0\le s<2j+i-l}$, $\{\alpha^{(i)}(t)\}_{i=0}^{\min\{k+j,2j+1\}}$, and symplectic products with positive weights of pairs of sections from $S_{k+j-1}\times S_{k+j}$. The monomials of Φ_j are quadratic w.r.t. the functions $\{\beta_i^{(s)}(t)\}_{1\le i\le l+1, 0\le s<k+j}$ and $\{\alpha^{(i)}(t)\}_{i=0}^{\min\{k+j,2j+1\}}$. By (3.25), these monomials are nonzero if and only if their coefficients have nonnegative weight. Besides the weights of these monomials are equal to

$$\deg \sigma(\widetilde{f}^{(k+j-1)}(t), \widetilde{f}^{(k+j)}(t)) = 2j+1.$$

Therefore, if $\beta_i^{(s)}(t)$ is contained in one of such nonzero monomials, then $\deg \beta_i^{(s)}(t)$ is not greater than $2j+1$. From this and (3.24) it follows that

$$s \le 2j+i-l \tag{3.28}$$

If we consider the monomial of $\sigma(\widetilde{f}^{(k+j-1)}(t), \widetilde{f}^{(k+j)}(t))$ containing the maximal possible derivative of the function $\beta_i(t)$, i.e. $\beta_i^{(2j+i-l)}(t)$, then the weight of the other factors in this monomial has to be equal to 0. Besides, in the expression of $\widetilde{f}^{(k+j)}(t)$ the function $\beta_i^{(2j+i-l)}(t)$ appears only near $e^{(k+l-j-1)}(t)$, while in the expression of $\widetilde{f}^{(k+j-1)}(t)$ it appears only near $e^{(k+l-j-2)}(t)$. Note that the only pair of sections in $S_{k+j-1}(t)\times\{e^{(k+l-j-1)}(t)\}$ having symplectic product of weight 0 is the pair $\big(f^{(k+j-1)}(t), e^{(k+l-j-1)}(t)\big)$ and the only pair of sections in $\{e^{(k+l-j-2)}(t)\}\times S_{k+j}(t)$ having symplectic product of weight 0 is $\big(e^{(k+l-j-2)}(t), f^{(k+j)}(t)\big)$. In the both cases the symplectic products are equal to $(-1)^j$ and the additional factor in the corresponding monomials is equal to $\alpha(t)$ (coming from the term $\alpha(t)f^{(k+j-1)}(t)$ in the expression of $\widetilde{f}^{(k+j-1)}(t)$ and from the term $\alpha(t)f^{(k+j)}(t)$ in the expression of $\widetilde{f}^{(k+j)}(t)$ respectively). From the Leibnitz rule it is not difficult to get from here that in the expression of $\sigma(\widetilde{f}^{(k+j-1)}(t), \widetilde{f}^{(k+j)}(t))$ in terms the moving frame associated with thew pair (e,f) the monomial, containing $\beta_i^{(2j+i-l)}(t)$, has the form

$$(-1)^j\alpha(t)\left(\binom{k+j-1}{2j+i-l}+\binom{k+j}{2j+i-l}\right)\beta_i^{(2j+i-l)}(t). \tag{3.29}$$

This completes the proof of the lemma. □

Let $\mathfrak{B}$ be a $(l+1)\times(l+1)$-matrix such that $\mathfrak{B}_{j+1,i}=\binom{k+j-1}{2j+i-l}+\binom{k+j}{2j+i-l}$ for $0\le j\le l$, $1\le i\le l+1$. For any $0\le p\le \left[\frac{l}{2}\right]$ let $\mathfrak{B}_p$ be a $(l+1-2p)\times(l+1-2p)$-matrix obtained from $\mathfrak{B}$ by erasing the last $2p$ columns, the first p rows, and the last p rows.

Lemma 4. a) $\det\mathfrak{B}_p\ne 0$ *for any* $0\le p\le\left[\frac{l}{2}\right]$;

[1]Binomial coefficients $\binom{n}{k}$ with $n<k$ are supposed to be equal to zero in (3.27).

b) *The system of differential equations* (3.27) *with* $0 \leq j \leq l$ *w.r.t. the functions* $\{\beta_i(t)\}_{i=1}^{l+1}$ *is equivalent to the system of equations of the following form*

$$(3.30)\quad \beta_{l+1-2r}^{(2l-4r+1)}(t) = \frac{1}{\alpha(t)}\Psi_r\left(\{\beta_{l+1-2j}^{(s)}(t)\}_{0\leq j\leq [l/2], 0\leq s\leq 2l-2j-\max\{2j,2r-1\}}, \alpha(t)\right), \quad 0 \leq r \leq [l/2],$$

$$(3.31)\quad \beta_{l-2r}(t) = \frac{1}{\alpha(t)}\Theta_r\left(\{\beta_{l+1-2j}^{(s)}(t)\}_{0\leq j\leq [l/2], 0\leq s\leq 2r+1-2j}, \alpha(t)\right), \quad 0 \leq r \leq [(l-1)/2]$$

where Ψ_r *and* Θ_r *are polynomial expressions w.r.t. their arguments such that the weights of each of their monomials are equal to* $2l-2r+1$ *and* $2r+1$ *respectively.*

Proof. **a)** First, by direct computations,

$$\binom{k+j-1}{2j+i-l} + \binom{k+j}{2j+i-l} = \frac{2k-i+l}{k+j}\binom{k+j}{2j+i-l}. \tag{3.32}$$

Let $\mathfrak{M}^{k,l}$ be a $(l+1)\times(l+1)$-matrix such that its $(j+1,i)$th entry is equal to $\binom{k+j}{2j+i-l}$ or, equivalently,

$$\left(\mathfrak{M}^{k,l}\right)_{j,i} = \binom{k+j-1}{2j+i-l-2}. \tag{3.33}$$

Similarly to above, for any $0 \leq p \leq \left[\frac{l}{2}\right]$ let $\mathfrak{M}_p^{k,l}$ be a $(l+1-2p)\times(l+1-2p)$-matrix obtained from $\mathfrak{B}$ by erasing the last $2p$ columns, the first p rows, and the last p rows. Since in the factor $\frac{2k-i+l}{k+j}$ in the formula (3.32) the indices i and j are separated, it is sufficient to prove the statement a) of the lemma for the matrices $\mathfrak{M}_p^{k,l}$ instead of $\mathfrak{B}_p$. Besides, by definition, it is not hard to see that

$$\mathfrak{M}_p^{k,l} = \mathfrak{M}^{k+p,l-2p}. \tag{3.34}$$

Denote

$$c(k,l) = \frac{(-1)^l l!}{(2l-1)!!(2l+1)!}\prod_{r=1}^{l}(2k+2r-1)\prod_{r=0}^{l}(k+r). \tag{3.35}$$

We will prove that

$$\mathfrak{M}^{k,l} = c_{k,l}\mathfrak{M}^{k+1,l-2} \tag{3.36}$$

The last formula will imply that $\mathfrak{M}^{k,l} = \prod_{r=0}^{[\frac{l}{2}]} c_{k+r,l-2r}$, which together with (3.34) and (3.35) will imply the statement a) of the lemma. Formula (3.36) will follow in turn from the following statement

Statement 1. *For any* $s \in \{1,\dots l\}$ *the last column of the matrix* $\mathfrak{M}^{k,l}$ *can be transformed by a series of elementary matrix transformations such that the* $(j,l+1)$*th entry of the transformed matrix is equal to*

$$d(k,s,j) = \frac{(-1)^s}{(2s-1)!!(2j-1)!}\prod_{r=1}^{s}(j-r)(2k+2r-1)\prod_{r=1}^{2j-s-1}(k-j+s+r) \tag{3.37}$$

(while all other columns remain as in $\mathfrak{M}^{k,l}$*).*

In particular, $d(k,s,j)=0$ for $1\le j\le s$ and $d(k,l,l+1)=c(k,l)$. These facts together with (3.34) will easily imply (3.36).

So, to complete the proof of the statement a) of Lemma 4 it remains to prove Statement 1. We do it by induction w.r.t. s. To get Statement 1 for $s=1$ we subtract the lth column multiplied by k from the $(l+1)$th column. Namely by direct computations, one has

$$\big(\mathfrak{M}^{k,l}\big)_{j,l+1}-k\big(\mathfrak{M}^{k,l}\big)_{j,l}=d(k,1,j).$$

Now assume that Statement 1 is true for $s=s_0<l$ and prove it for $s=s_0+1$. We work with the matrix transformed from the matrix $\mathfrak{M}^{k,l}$ as in Statement 1 for $s=s_0$. Using (3.34) and induction hypothesis of Statement 1, applied for the matrix $\mathfrak{M}^{k+1,l-2}$ and $s=s_0-1$, we will get that the $(l-1)$th row of the matrix $\mathfrak{M}^{k,l}$ can be transformed by a series of elementary matrix transformations such that its $(j,l-1)$th entry is equal to $d(k+1,s_0-1,j-1)$ for all $2\le j\le l$, while the $(1,l-1)$th entry of the transformed matrix is equal to 0, because $\big(\mathfrak{M}^{k,l}\big)_{1,i}=0$ for all $1\le i\le l-1$. After all these transformations, in order to get Statement 1 for $s=s_0+1$ we add the $(l-1)$th column of the obtained matrix multiplied by

$$-\frac{d(k,s_0,s_0+1)}{d(k+1,s_0-1,s_0)}=\frac{(2k+1)k}{2(4s_0^2-1)}$$

to its $(l+1)$th row. Namely, by direct calculations, one has the following identity

$$d(k,s_0,j)+\frac{(2k+1)k}{2(4s_0^2-1)}\,d(k+1,s_0-1,j-1)=d(k,s_0+1,j),$$

which implies Statement 1 for $s=s_0+1$ (note that the transformations we made with the $(l-1)$th column can be turned back). With this the proof of statement a) of Lemma 4 is completed

b) Consider the system of differential equations (3.27) with $0\le j\le l$ w.r.t. the functions $\{\beta_i(t)\}_{i=1}^{l+1}$. If for given $j_1<j_2$ we differentiate $2(j_2-j_1)$ times equation (3.27) with $j=j_1$ and then subtract the obtained equation multipliead by a constant from equation (3.27) with $j=j_2$, then we obtain a system of equations

(3.38)

$$\alpha(t)\sum_{i=1}^{l+1}(\widetilde{\mathfrak{B}})_{j,i}\beta_i^{(2j+i-l)}(t)=\widetilde{\Phi}_j\left(\{\beta_i^{(s)}(t)\}_{1\le i\le l+1,0\le s<2j+i-l},\{\alpha^{(i)}(t)\}_{i=0}^{\min\{k+j,2j+1\}}\right),\quad 0\le j\le l,$$

where the matrix $\widetilde{\mathfrak{B}}$ is obtained from the $\mathfrak{B}$ by subtraction of j_1th row multiplied by the same constant from the j_2th rows of the latter and the functions $\widetilde{\Phi}_j$ have the properties similar to the properties of functions Φ_j from Lemma 3. Therefore the system (3.27) with $0\le j\le l$ is equivalent to the system of the type (3.38) with matrix $\widetilde{\mathfrak{B}}$ obtained from the matrix $\mathfrak{B}$ by a series of elementary transformations with its rows. In this way we apply the Gauss algorithm first to the $(l-2p)$th column of $\mathfrak{B}$ killing all entries below the $(p+1,l-2p)$th entry step by step starting with $p=0$ and ending up with $p=\left[\frac{l-1}{2}\right]$. This is possible, because $(\mathfrak{B})_{p+1,l-2p}=1$. Further, from the statement a) of the lemma it follows that we can apply the Gauss algorithm to the $\left(2\left\{\frac{l-1}{2}\right\}+2p\right)$th column killing all entries below the $\left(\left[\frac{l+3}{2}\right]+p,2\left\{\frac{l-1}{2}\right\}+2p\right)$th entry step by step starting with $p=0$ and ending up with $p=\left[\frac{l}{2}\right]$. Besides the $\left(\left[\frac{l+3}{2}\right]+p,2\left\{\frac{l-1}{2}\right\}+2p\right)$th entries of the matrix, obtained in this way, are not equal to zero. Therefore, the system of equations (3.27) is equivalent to the following one:

(3.39)

$$\beta_{l-2r_1}(t)=\frac{1}{\alpha(t)}\widetilde{\Theta}_r\left(\{\beta_{l+1-2j}^{(2(r_1-j)+1)}(t)\}_{j=0}^{r_1},\{\beta_i^{(s)}(t)\}_{1\le i\le l+1,0\le s<2r_1+i-l},\{\alpha^{(i)}(t)\}_{i=0}^{2r_1+1}\right),$$

$$\beta_{l+1-2r_2}^{(2l-4r_2+1)}(t)=\frac{1}{\alpha(t)}\widetilde{\Psi}_r\left(\{\beta_{l+1-2j}^{(2l-2(r_2+j)+1)}(t)\}_{j=0}^{r_2-1},\{\beta_i^{(s)}(t)\}_{1\le i\le l+1,0\le s\le l-2r_2+i},\{\alpha^{(i)}(t)\}_{i=0}^{2l-2r_2+1}\right),$$

$$0\le r_1\le\left[\frac{l-1}{2}\right],\quad 0\le r_2\le\left[\frac{l}{2}\right],$$

Substitute the righthand side of the first relation of (3.39) with $r_1=1$ instead of $\beta_l(t)$ into all other equations, then substitute the righthand side of the first relation (3.39) with $r_1=2$ instead of $\beta_{l-2}(t)$ into relations with $r_1>2$ and any admissible r_2, and so on up to $r_1=\left[\frac{l-1}{2}\right]$. Then in all obtained expressions substitute $\alpha'(t)$ and the higher derivatives of $\alpha(t)$ from the equation (3.22), then repeat this procedure recursively until all derivatives of $\alpha(t)$ will be replaced. In this way we get equations (3.31) for any admissible r and the equation (3.30) for $r=\left[\frac{l}{2}\right]$. Further, we substitute the righthand side of the second relation of (3.39) with $r_2=\left[\frac{l}{2}\right]$ instead of $\beta_1'(t)$ for even l and instead of $\beta_2^{(3)}(t)$ for odd l into the remaining equations of (3.39), then substitute the righthand side of the second relation of (3.39) with $r_2=\left[\frac{l}{2}\right]-1$ instead of $\beta_1^{(5)}(t)$ for even l and instead of $\beta_2^{(7)}(t)$ for odd l into the remaining equations of (3.39), and so on. Then as before in all obtained expressions substitute $\alpha'(t)$ and the higher derivatives of $\alpha(t)$ from the equation (3.22), then repeat this procedure recursively until all derivatives of $\alpha(t)$ will be replaced. In this way we get equations (3.30) for all remaining admissible r. The proof of the statement b) of Lemma 4 is completed □

The statement of Theorem 1 immediately follows from the system of the differential equations w.r.t. $\alpha(t)$ and $\beta_{l+1-2r}(t)$ with $0\le r\le[l/2]$, consisting of equation (3.22) and all equations (3.30). □

As in the case of the rectangular diagram the curve (2.8) (or equivalently the curve γ) can be endowed with the canonical projective structure, but its construction depends on vanishing or nonvanishing of certain relative invariant of the curve.

Fix a symplectic form σ from the one-parametric family of symplectic forms on $\Delta(\gamma)$ and a parametrization $\varphi:\gamma\mapsto\mathbb{R}$ of γ. Let $t=\varphi(\lambda)$. To define the mentioned relative invariant note that by Proposition 3 we have $e^{(i)}(t)\in J^{(-k-l+i+1)}\big(\varphi^{-1}(t)\big)$. Moreover,

$$J^{(-k-l+i+1)}\big(\varphi^{-1}(t)\big)=\operatorname{span}\{e^{(j)}(t)\}_{j=0}^{i}\quad\forall\,0\le i\le l-1. \tag{3.40}$$

Since spaces $J^{(i)}(\lambda)$ are isotropic for $i<0$ and $J^{(-1)}=(J^{(0)})^{\angle}$, we get $\sigma\big(e^{(i)}(t),e^{(i+1)}(t)\big)=0$ for all $0\le i\le k+l-2$. On the other hand, the quantity $\sigma\big(e^{(k+l-1)}(t),e^{(k+l)}(t)\big)$ is not necessary equal to 0. Set

$$I_0(\lambda)\stackrel{def}{=}\sigma\big(e^{(k+l-1)}(t),e^{(k+l)}(t)\big),$$

where $t=\varphi(\lambda)$. The quantity $I_0(\lambda)$ depends on a choice of a nonzero section of $J^{(-k-l+1)}$, a parametrization of γ, and a form σ but it is just multiplied by a positive scalar when one goes from one such choice to another one. So, the quantity $I_0(\lambda)$ is a well defined relative invariant of a curve (2.8) at λ, i.e. $I_0(\lambda)$ is either zero or nonzero, independently of a choice of a nonzero section of $J^{(-k-l+1)}$ and a parametrization of γ (moreover, its sign is preserved as well). Our construction of a canonical projective structure is different in the cases $I_0\neq 0$ and $I_0\equiv 0$. However, as we will see later, this branching in the construction of a canonical projective structure does not make any influence on the constancy of the symbol of the obtained frame bundles and therefore on the prolongation procedure for them.

a) Canonical projective structure in the case $I_0 = 0$. Given a normal pair of sections (e, f) associated with a parametrization φ of γ and the symplectic form σ let

$$\rho_{2,\varphi}(\lambda) = \sigma\big(e^{(k+l)}(t), f^{(k)}(t)\big), \quad t = \varphi(\lambda). \tag{3.41}$$

Note that $\rho_{2,\varphi}(\lambda)$ does not depend on a choice of normal pair (e, f) . The transformation rule of $\rho_{2,\varphi}$ under a reparametrization of γ is similar to (3.16) Indeed, let φ_1 be another parametrization and $\upsilon = \varphi \circ \varphi_1^{-1}$. Then it is not hard to show that $\rho_{2,\varphi}$ and ρ_{2,φ_1} are related as follows:

$$\rho_{2,\varphi_1}(\lambda) = \upsilon'(\tau)^2 \rho_{2,\varphi}(\lambda) + C^2_{k,l}\mathbb{S}(\upsilon)(\tau), \quad \tau = \varphi_1(\lambda)$$

where, as before, $\mathbb{S}(\upsilon)$ is the Schwarzian derivative of υ and $C^2_{k,l}$ is a nonzero constant.

The set of all parametrizations φ of γ such that

$$\rho_{2,\varphi} \equiv 0 \tag{3.42}$$

defines the canonical projective structure on γ.

b) Canonical projective structure in the case $I_0 \neq 0$. As a matter of fact in this case there is much more simple way to construct distinguished moving frames for a curve of flags (2.8). Indeed, given a parametrization φ and a symplectic form σ there exists a unique section $\bar{e}$ of $J^{(-k-l+1)}$ such that

$$|\sigma\big(\bar{e}^{(k+l-1)}(t), \bar{e}^{(k+l)}(t)\big)| \equiv 1,$$

i.e. the absolute value of the relative invariant I_0 for such choice of section of $J^{(-k-l+1)}$. is equal to 1.

Lemma 5. *Among all sections of $J^{(-k+1)}$ there is a unique section $\bar{f}$ such that $(\bar{e}, \bar{f})$ is a principal pair of section of the curve* (2.8) *and the following relations hold:*

$$\sigma\big(\bar{f}^{(k-1)}(t), \bar{e}^{(k+l-1)}(t)\big) \equiv 1, \tag{3.43}$$

$$\sigma\big(\bar{f}^{(k-1+j)}(t), \bar{e}^{(k+l)}(t)\big) = 0 \quad \forall\, 0 \le j \le l \tag{3.44}$$

Proof. Take a section $\hat{f}$ of the bundle $J^{(-k+1)}$ such that $(\bar{e}, \hat{f})$ is a principal pair of section of the curve (2.8). The condition $\sigma\big(\hat{f}^{(k-1)}(t), \bar{e}^{(k+l-1)}(t)\big) \equiv 1$ defines $\hat{f}$ modulo $\big(J^{(-k)}\big)^{(1)}$. Further, taking into account that $I_0 \neq 0$, it is easy to see that the condition $\sigma\big(\hat{f}^{(k-1)}(t), \bar{e}^{(k+l)}(t)\big) = 0$ defines $\hat{f}$ modulo $J^{(-k)}$. More generally, the conditions

$$\sigma\big(\hat{f}^{(k-1+j)}(t), \bar{e}^{(k+l)}(t)\big) = 0,$$

for $0 \le j \le i$, defines $\hat{f}$ modulo $J^{(-k-i)}$. Our lemma follows from the fact that $J^{(-k-l)} = 0$. □

Take the section $\bar{f}$ from the previous lemma and let

$$\rho_{3,\varphi}(\lambda) = \sigma\big(\bar{f}^{(k-1)}(t), \bar{f}^{(k)}(t)\big), \quad t = \varphi(\lambda). \tag{3.45}$$

The transformation rule for $\rho_{3,\varphi}$ under a reparametrization of γ is similar to (3.16). Indeed, let φ_1 be another parametrization and $\upsilon = \varphi \circ \varphi_1^{-1}$. Then it is not hard to show that $\rho_{3,\varphi}$ and ρ_{3,φ_1} are related as follows:

$$\rho_{3,\varphi_1}(\lambda) = \upsilon'(\tau)^2 \rho_{3,\varphi}(\lambda) + C^3_{k.l}\mathbb{S}(\upsilon)(\tau), \quad \tau = \varphi_1(\lambda)$$

where, as before, $\mathbb{S}(\upsilon)$ is the Schwarzian derivative of υ and $C^3_{k,l}$ is a nonzero constant. *The set of all parametrizations φ of γ such that*

$$\rho_{3,\varphi} \equiv 0 \tag{3.46}$$

defines the canonical projective structure on γ.

Now let, as before, N be a manifold of all leaves of the characteristic foliation $\mathcal{C}$ in a small neighborhood of a point $\lambda_0 \in \mathbb{P}D^\perp \backslash \mathbb{P}(D^2)^\perp$ such that the linearizations of the flag $\{\mathcal{J}^{(i)}\}$ along any leaf of $\mathcal{C}$ in this neighborhood has the Young diagram T of type (k,l), where $l>0$. Consider a fiber bundle $P_{k,l}$ over N such that its fiber over a point γ is a set of tuples $\big(\gamma,\varphi,\sigma,(e,f)\big)$, where φ is a projective parametrization of the leaf (the abnormal extremal) γ , σ is a symplectic form σ from the one-parametric family of forms on $\Delta(\gamma)$, and (e,f) is a germ of normal pair (e,f) of sections associated with the parametrization φ and the form σ. In contrast to the bundles $P_{k,0}$, this bundle has no structure of a principal bundle. On the other hand, on each fiber of this bundle there is a distinguished global frame. The vector fields of this frame play the role of fundamental vector fields in the case of principle bundle. They do not constitute a basis of a Lie algebra but this fact is not important for our further constructions.

Let us construct these vector fields. Note that each fiber of $P_{k,l}$ over a point λ_0 is foliated by a corank 3 foliation Fol such that each its leaf corresponds to a fixed projective parametrization φ such that $vf^{-1}(0)\in N$. First we will construct a global moving frame on each leaf of this foliation. For this fix a point $\Gamma_0\in\mathbb{P}_{k,l}$, $\Gamma_0=\big(\gamma,\varphi,\sigma,(e,f)\big)$. For any $0\le\bar r\le\left[\frac{l}{2}\right], 0\le\bar i\le 2l-4\bar r$ let $s\to\Gamma_{r,i}(s)$ be a curve on the fiber of $P_{k,l}$ over γ such that the point $\Gamma_{r,i}(s)$ corresponds to the parametrization φ and the symplectic form σ but the germ of normal pair $(\widetilde e,\widetilde f)$, corresponding to $\Gamma_{r,i}(s)$, is related to the pair (e,f) by (3.21) with the the functions $\alpha,\beta_1,\dots,\beta_{l+1}$ satisfying

$$\alpha(0)=1,\quad \beta^{(\bar i)}_{l+1-2\bar r}(0)=\delta_{i,\bar i}\delta_{r,\bar r}s,\quad \forall\, 0\le\bar r\le\left[\frac{l}{2}\right], 0\le\bar i\le 2l-4r, \tag{3.47}$$

where $\delta_{i,j}$ is the Kronecker index. It is clear that $\Gamma_{r,i}(0)=\Gamma_0$. Define the vector field $\mathcal{P}_{r,i}$ as follows: $\mathcal{P}_{r,i}(\Gamma_0)$ is the velocity of the curve $\Gamma_{r,i}$ at $s=0$. Further, let $s\to\Xi_1(s)$ be a curve on the fiber of $P_{k,l}$ over γ such that the point $\Xi_1(s)$ corresponds to the parametrization φ, the symplectic form σ, and the germ of normal pair $(\exp^s e,\exp^{-s} f)$ associated with the parametrization φ and the form σ. Obviously, $\Xi_1(0)=\Gamma_0$. Define a vector field $\mathcal{Z}_1$ such that $\mathcal{Z}_1(\Gamma_0)$ is the velocity of the curve Ξ_1 at $s=0$. Finally, let $s\to\Xi_2(s)$ be a curve on the fiber of $P_{k,l}$ over γ such that the point $\Xi_2(s)$ corresponds to the parametrization φ, the symplectic form $\sigma_s=\exp^{-2s}\sigma$, and the germ of normal pair $(\exp^s e,\exp^s f)$ associated with the parametrization φ and the form σ_s. By construction, $\Xi_2(0)=\Gamma_0$. Define a vector field $\mathcal{Z}_2$ such that $\mathcal{Z}_2(\Gamma_0)$ is the velocity of the curve Ξ_1 at $s=0$. The tuple of vector fields $\big(\{\mathcal{P}_{r,i}\}_{0\le r\le[\frac{l}{2}],0\le i\le 2l-4r},\mathcal{Z}_1,\mathcal{Z}_2\big)$ constitute the global frame on each leaf of the foliation Fol.

To complete it to a frame on the fibers of $P_{k,l}$ first note that if a point $\Gamma_0=\big(\gamma,\varphi,\sigma,(e,f)\big)$ lies in $P_{k,l}$ then the points $\Gamma(s)=\big(\gamma,\varphi(\cdot)-s,\sigma,(e,f)\big)$ belong to $P_{k,l}$ (here we replace the parametrization φ by its shift $\varphi(\cdot)-s$ for a constant s). Define a vector field $\mathcal{X}$ such that $\mathcal{X}(\Gamma_0)$ is the velocity of the curve $s\to\Gamma(s)$ at $s=0$. Further there exists a natural action of the group of real Möbius transformations preserving 0 ($\sim \mathrm{ST}(2,\mathbb{R})$) on the fibers of $P_{k,l}$. Indeed, take first a Möbius transformations υ preserving 0 and ,as before, take a point $\Gamma_0\in P_{k,l}$ such that $\Gamma_0\sim\big(\gamma,\varphi,\sigma,(e,f)\big)$. By direct computations, one can show that if the pair $(e(\lambda),f(\lambda))$, $\lambda\in\gamma$, is a normal pair of sections associated with the parametrization φ and the symplectic form σ, then the pair

$$\big(e_\upsilon(\lambda),f_\upsilon(\lambda)\big)=\big(\upsilon'(\tau)^{-(k+l/2+1)}e(\lambda),\upsilon'(\tau)^{-(k+l/2+1)}f(\lambda)\big)$$

with $\tau = \upsilon^{-1}\circ\varphi(\lambda)$, $\lambda\in\gamma$, is a normal pair of sections associated with the parametrization $\upsilon^{-1}\circ\varphi$ and the symplectic form σ. Then we set that υ acts on the fiber of $P_{k,l}$ over λ_0 by sending Γ_0 to the point $\big(\lambda_0, \upsilon^{-1}\circ\varphi, \sigma, (e_\upsilon, f_\upsilon)\big)$. This defines the action of the group $\mathrm{ST}(2,\mathbb{R})$ on the fibers of $P_{k,l}$. Then any choice of a basis (H, Y) of the corresponding Lie algebra $st(2,\mathbb{R})$ defines two more vector fields $\mathcal{H}$ and $\mathcal{Y}$ on the fibers of $P_{k,l}$ which together with the vector field $\mathcal{X}$ complete the tuple $\big(\{\mathcal{P}_{r,i}\}_{0\leq r\leq[\frac{l}{2}],0\leq i\leq 2l-4r}, \mathcal{Z}_1, \mathcal{Z}_2\big)$ to the frame on these fibres.

Now to any point $\Gamma\in P_{k,l}$, where $\Gamma = \big(\gamma,\varphi,\sigma,(e,f)\big)$, assign the frame $\mathfrak{F}_2(\Gamma)$ on $\Delta(\gamma)$ at the point $\varphi^{-1}(0)$ of normal moving frames of the curve (2.8) generated by the pair (e,f) and associated with the parametrization φ and symplectic form σ. In contrast to the case of rectangular diagram, we cannot claim that the mapping $\mathfrak{F}_2$ are injective but we have the following

Proposition 4. *The mapping $\mathfrak{F}_2$ is an immersion.*

Proof. Let, as in the Introduction, V be a vector spaces, endowed with the filtration vector (1.1) and with the distinguished basis $(e_1,\ldots,e_{2k+l-1}, f_1,\ldots,f_{2k+l-1})$ satisfying conditions (1) and (2) of subsection 1.1 b). Then any frame Υ on $\Delta(\gamma)$ can be identified with the isomorphism $\widehat{\Upsilon}: V\to\Delta(\gamma)$ sending the distinguished frame of V to the frame Υ. Further, any vector A belonging to the tangent space at Υ to the set of all frames on $\Delta(\gamma)$ can be naturally identified with an element $\mathcal{I}_A$ of $\mathfrak{gl}(V)$. Indeed, if $s\to\Upsilon(s)$ is a smooth curve of frames on $\Delta(\gamma)$ such that $\Upsilon(0)=\Upsilon$ and $\Upsilon'(0)=A$ then let $\mathcal{I}_A = \widehat{\Upsilon}^{-1}\circ\frac{d}{ds}\widehat{\Upsilon(s)}|_{s=0}$. So, any vector field B on $P_{k,l}$ tangent to its fibers defines the mapping $\mathcal{I}_B$ from $P_{k,l}$ to $\mathfrak{gl}(V)$ which sends a point $\Gamma_0\in P_{k,l}$ to the operator $\mathcal{I}_{d\widetilde{\mathfrak{F}}_3 B(\Gamma_0)}$.

From the constructions it is easy to see that for the vector fields $\mathcal{Z}_1$, $\mathcal{Z}_2$, $\mathcal{X}$, $\mathcal{H}$, and $\mathcal{Y}$ the corresponding mappings $\mathcal{I}_{\mathcal{Z}_1}$, $\mathcal{I}_{\mathcal{Z}_2}$, $\mathcal{I}_{\mathcal{X}}$, $\mathcal{I}_{\mathcal{H}}$, and $\mathcal{I}_{\mathcal{Y}}$ are constant, i.e. do not depend on points of $P_{k,l}$. This is not the case for the mappings corresponding to the vector fields $\mathcal{P}_{r,i}$. On the other hand, the filtration on V induces a natural filtration on $\mathfrak{gl}(V)$, i.e. a nondecreasing (by inclusion) sequence of subspaces $\mathfrak{gl}(V)^{(i)}$ of $\mathfrak{gl}(V)$ such that

$$\mathfrak{gl}(V)^{(i)} = \{\widehat{A}\in\mathfrak{gl}(V):\ \text{if } v\in V^{(j)} \text{ then } \widehat{A}v\in V^{(j+i)}\}. \tag{3.48}$$

We say that the operator $\widehat{A}\in\mathfrak{gl}(V)$ has *the weight (or degree) equal to i* if $\widehat{A}$ is in $\mathfrak{gl}(V)^{(i)}$ but not in $\mathfrak{gl}(V)^{(i-1)}$

Lemma 6. *The operators $\mathcal{I}_{\mathcal{P}_{r,i}(\Gamma)}$ corresponding to the vector field $\mathcal{P}_{r,i}$ have weight equal to $-2r-i$. The equivalence class of the operators $\mathcal{I}_{\mathcal{P}_{r,i}(\Gamma)}$ in $\mathfrak{gl}(V)^{(-2r-i)}/\mathfrak{gl}(V)^{(-2r-i-1)}$ does not depend on $\Gamma\in P_{k,l}$.*

Proof. Take two points $\Gamma_0,\Gamma\in P_{k,l}$, where $\Gamma_0 = \big(\gamma,\varphi,\sigma,(e,f)\big)$ and $\Gamma = \big(\gamma,\varphi,\sigma,(\widetilde{e},\widetilde{f})\big)$. Let $(e_1(t),\ldots,e_{2k+l-1}(t), f_1(t),\ldots f_{2k+l-1}(t))$ and $(\widetilde{e}_1(t),\ldots,\widetilde{e}_{2k+l-1}(t),\widetilde{f}_1(t),\ldots\widetilde{f}_{2k+l-1}(t))$ be the corresponding moving frames over γ, where $t=\varphi(\lambda)$, $\lambda\in\gamma$. Namely, $e_i(t) = e^{(i-1)}(t)$, $\widetilde{e}_i(t) = \widetilde{e}^{(i-1)}(t)$, $f_i(t) = f^{(i-1)}(t)$, and $\widetilde{f}_i(t) = \widetilde{f}^{(i-1)}(t)$. The pairs (e,f) and $(\widetilde{e},\widetilde{f})$ are related by (3.21) for some functions α, $\beta_1,\ldots,\beta_{l+1}$, where $\alpha(0)=1$.

Let us study how the the vectors $\widetilde{f}_j(t)$ are expressed by the frame $(e_1(t),\ldots,e_{2k+l-1}(t), f_1(t),\ldots f_{2k+l-1}(t))$. For this first note that $e'_{2k+l-1}(t)\in J^{(k-1)}\big(\varphi^{-1}(t)\big)$. Indeed, from the condition (1) of the definition of the (k,l)- quasisymplectic frame it follows that $\sigma\big(e_i(t), e_{2k+l-1}(t)\big)=0$ for any $1\leq i\leq l+1$. Then by differentiation $\sigma\big(e_i(t), e'_{2k+l-1}(t)\big)=0$ for any $1\leq i\leq l$. Recalling that $J^{(-k)}\big(\varphi^{-1}(t)\big) = \mathrm{span}\{e_j(t)\}_{j=1}^{l}$ (see (3.40)) and that $(J^{(-k)})^{\angle}\big(\varphi^{-1}(t)\big) = J^{(k-1)}\big(\varphi^{-1}(t)\big)$ by (2.7) we get $e'_{2k+l-1}(t)\in J^{(k-1)}$. It implies in turn that $e^{(j)}_{2k+l-1}(t)\in J^{(k+j-2)}$.

Assume that for $1 \leq j \leq l$

$$e^{(j)}_{2k+l-1}(t) = \sum_{p=1}^{2k+l-1} \xi_{jp}(t)e_p(t) + \sum_{p=1}^{2k-2+j} \zeta_{jp}(t)f_p(t). \tag{3.49}$$

In order to make both sides of (3.49) to be homogeneous of the same degree, in addition to weights defined above, let us define weights (or degrees) of functions $\xi_{ji}(t)$ and $\zeta_{ji}(t)$ and their derivatives as follows:

$$\deg \xi^{(s)}_{jp}(t) = s + 2k + l + j - p - 1, \quad \deg \zeta^{(s)}_{jp}(t) = s + 2k + j - p - 1. \tag{3.50}$$

Differentiating (3.21) $j-1$ times and making, if necessary, appropriate substitutions from (3.49), we get

$$\widetilde{f_j}(t) = \alpha(t)f_j(t) + \sum_{p=1}^{j-1} \mu_{jp}(t)f_p(t) + \sum_{p=1}^{2k+l-1} \lambda_{jp}(t)e_p(t)$$

where functions λ_{jp} are polynomial expressions with constant coefficients w.r.t. the functions $\beta_s(t)$, $\xi_{sp}(t)$ and their derivatives, functions $\mu_{jp}(t)$ are polynomial expressions with constant coefficients w.r.t. the derivatives of function $\alpha(t)$ and functions $\beta_s(t)$, $\zeta_{sp}(t)$ and their derivatives. In all these expressions substitute all functions $\beta_{l-2r}(t)$ (and their derivatives) by the righthand sides of (3.31) (and their derivatives). Then substitute (also recursively, if necessary) all functions $\beta^{(2l-4r+1)}_{l+1-2r}(t)$ (and their derivatives) by the righthand sides of (3.30) (and their derivatives) and all derivatives of $\alpha(t)$ by the righthand sides of (3.22) (and their derivatives). After all these substitutions we will get finally that all function $\lambda_{jp}(t)$ and $\mu_{jp}(t)$ are polynomial expressions w.r.t. the functions

$$\{\beta^{(i)}_{l+1-2r}(t)\}_{0\leq r\leq[\frac{l}{2}],0\leq i\leq 2l-4r}\ , \tag{3.51}$$

such that the coefficients of their monomials are in turn polynomial expressions with universal constant coefficients [2] w.r.t. symplectic products with positive weights of pairs of sections from the set $\Big\{\{e_s(t)\}_{s=1}^{2k+l-1}, \{f_s(t)\}_{s=1}^{2k+l-1}\Big\}$ and functions α ξ_{s_1p}, ζ_{s_2p}, and their derivatives. Note that by our construction the weights of the functions ξ_{s_1p} and ζ_{s_2p} are positive. Therefore by comparison of weights the function $\beta^{(i)}_{l+1-2r}(t)$ cannot appear in the expression for $\lambda_{jp}(t)$ with

$$\deg e_p(t) - \deg \widetilde{f_j}(t) > -\deg \beta^{(i)}_{l+1-2r}(t) = -2r - i.$$

By (3.23) the latter is equivalent to the relation $p > j + l - 2r - i$. Further, for $p = j + l - 2r - i$ in the polynomial expression of $\lambda_{jp}(t)$ w.r.t. the tuple (3.51) take the coefficient of the monomial, containing only the function $\beta^{(i)}_{l+1-2r}(t)$ (and not its power or other functions from the tuple (3.51)), if it exists. Then the weight of this coefficient is equal to zero. Therefore this coefficient is a polynomial w.r.t. the function $\alpha(t)$ with universal coefficients [3]. Let U_{rij} be the value of this polynomial at $t = 0$. Note that this constant is again universal.

Besides, in the polynomial expression of $\mu_{jp}(t)$ w.r.t. the tuple (3.51) the coefficient of the monomial, containing only the function $\beta^{(i)}_{l+1-2r}(t)$, has positive weight, because each monomials

[2] By universality of the constants we mean that they are the same for any curve of flags with the fixed Young diagram.

[3] It can be shown that this coefficient is equal to $\alpha(t)$ multiplied by a constant.

of this coefficient contains either derivatives of $\alpha(t)$ or functions $\zeta_{sp}(t)$ or their derivatives. Hence the function $\beta^{(i)}_{l+1-2r}(t)$ cannot appear in the expression for $\lambda_{jp}(t)$ with $p \geq j+l-2r-i$.

All this implies that the operator $\mathcal{I}_{\mathcal{P}_{r,i}(\Gamma_0)}$ satisfies the following relation:

$$\mathcal{I}_{\mathcal{P}_{r,i}(\Gamma_0)}\big(f_j\big) = U_{rij}e_{j+l-2r-i} \mod \widetilde{V}^{(-k+j-2r-i-1)}. \tag{3.52}$$

By the similar arguments, one gets

$$\mathcal{I}_{\mathcal{P}_{r,i}(\Gamma_0)}\big(e_j\big) \in V^{(-k-l+j-2r-i-1)}. \tag{3.53}$$

Taking into account that $e_j \in V^{(-k-l+j)}$ and $f_j \in V^{(-k+j)}$, we obtain from (3.52)-(3.53) that $\mathcal{I}_{\mathcal{P}_{r,i}(\Gamma_0)} \in \mathfrak{gl}(V)^{(-2r-i)}$ and that the equivalence class of the operator $\mathcal{I}_{\mathcal{P}_{r,i}(\Gamma_0)}$ in the space $\mathfrak{gl}(V)^{(-2r-i)}/\mathfrak{gl}(V)^{(-2r-i-1)}$ does not depend on $\Gamma_0 \in P_{k,l}$. To complete the proof of the lemma it remains only to show that $\mathcal{I}_{\mathcal{P}_{r,i}(\Gamma_0)} \notin \mathfrak{gl}(\widetilde{V})^{(-2r-i-1)}$ or, equivalently, that given a pair (r,i) the constant U_{rij} does not vanish for at least one pair j. From universality of the constants U_{rij} it is sufficient to check only for one curve of flags with the given Young diagram. We shall consider the flat curve $\mathfrak{F}_{k,l}$, defined in the Introduction, as a simplest possible case.

Directly from the definition it follows that for the flat curve the functions $\xi_{jp}(t)$ and $\zeta_{jp}(t)$ from (3.49) vanish. Besides, in section 4, Theorem 2, it will be shown that for the flat curve any normal (quasi-symplectic) frame is symplectic. Therefore, symplectic products with positive weights of pairs of sections from the set $\Big\{\{e_s\}_{s=1}^{2k+l-1}, \{f_s\}_{s=1}^{2k+l-1}\Big\}$ vanishes as well. This implies that for the flat curve the operator $\mathcal{I}_{\mathcal{P}_{r,i}(\Gamma_0)}$ satisfies

$$\begin{cases} \mathcal{I}_{\mathcal{P}_{r,i}(\Gamma_0)}\big(e_j\big) = 0 \\ \mathcal{I}_{\mathcal{P}_{r,i}(\Gamma_0)}\big(f_j\big) = U_{rij}e_{j+l-2r-i}. \end{cases} \tag{3.54}$$

So, if $U_{rij} = 0$ for any j, then $\mathcal{I}_{\mathcal{P}_{r,i}(\Gamma_0)} = 0$, but the latter is impossible. Indeed, let Fol_1 be a subfoliation of Fol such that the points of the same leaf of Fol_1 correspond not only to the same projective parametrization of γ , but also to the same symplectic form σ from the one-parametric family of forms on $\Delta(\gamma)$ and the same first section $e_1(t)$ from the normal pair of sections. In the flat case equations (3.30) w.r.t. the functions $\{\beta_r(t)\}_{r=0}^{[l/2]}$ are linear. Therefore each leaf of the foliation Fol_1 has natural affine structure. The mapping $\mathfrak{F}_2$ sends the leaf of the foliation Fol_1 passing through the point Γ_0 to the set of frames on $\Delta(\gamma)$ of the type

$$\Big\{\{e_s(0)\}_{s=1}^{2k+l-1}, \{f_s(0) + \sum_{p=1}^{2k+l-1} \lambda_{sp}e_p(0)\}_{s=1}^{2k+l-1}\Big\},$$

which also has natural affine structure. Besides, it is clear that in the flat case the mapping $\mathfrak{F}_2$ is affine on each leaf of Fol_1. From the proof of Theorem 2 below the dimensions of the image of this leaf w.r.t. $\mathfrak{F}_2$ is equal to the dimension of this leaf. In particular, this implies that in the flat case the restriction of $\mathfrak{F}_2$ to each leaf of Fol_1 (and therefore $\mathfrak{F}_2$ itself) is an immersion. Since the tuple of vectors $\{\mathcal{P}_{r,i}(\Gamma_0)\}_{0\leq r\leq [\frac{l}{2}], 0\leq i\leq 2l-4r}$ span the tangent space at Γ_0 to the leaf of Fol_1 passing through Γ_0, we get from here that $\widetilde{I}_{P_{r,i}(\Gamma_0)} \neq 0$. This completes the proof of the lemma. □

The previous arguments also shows that the mapping $\mathfrak{F}_2$ is an immersion. □

4. Prolongation of filtered frame bundles on corank 1 distributions

4.1. **Graded skew-symmetric forms, symbols, and $W-$structures.** Collecting together the common features of both cases considered in the previous section, we arrive naturally to the following abstract setting.

Let $\mathcal{M}$ be a smooth manifold endowed with a bracket-generating distribution $\mathfrak{D}$ of corank 1. On each subspace $\mathfrak{D}(x)$, $x \in \mathcal{M}$, a skew-symmetric bilinear form ω_x is defined, up to a multiplication by a nonzero constant: $\omega_x = d\alpha(x)|_{\mathfrak{D}(x)}$, where α is a nonzero one form, annihilating the distribution $\mathfrak{D}$. Note that we do not assume that the distribution $\mathfrak{D}$ is contact so the form ω_x is not symplectic in general. As in the symplectic case, a subspace Λ of $\mathfrak{D}(x)$ is called isotropic (w.r.t. the form ω_x), if $\omega_x|_\Lambda = 0$. Also, given a subspace $\Lambda \subset \mathfrak{D}(x)$, denote by $\Lambda^\angle = \{v \in \mathfrak{D}(x) : \omega_x(v, \ell) = 0\ \forall \ell \in \Lambda\}$, the generalized skew-symmetric complement of Λ.

Further let V be a vector space of dimension $\dim \mathcal{M} - 1$ (equal to rank of $\mathfrak{D}$) endowed with a filtration

$$V = V^{(I)} \supseteq V^{(I-1)} \supseteq \ldots \supseteq V^{(-I-1)} \supseteq \ldots \supseteq V^{(-I_1)} = 0. \tag{4.1}$$

Also assume that V is endowed with a distinguished basis compatible with the filtration (4.1). Denote by $\mathrm{F}(\mathfrak{D})$ the bundle over $\mathcal{M}$ of all frames of $\mathfrak{D}$. It can be identified with the set of all isomorphisms $\phi_x \colon V \to \mathfrak{D}(x)$, $x \in \mathcal{M}$: to any frame $\mathfrak{D}(x)$ one assigns the isomorphism ϕ_x, which sends the distinguished basis of V to this frame. Further, let $\mathrm{F}_V(\mathfrak{D})$ be the subbundle of $\mathrm{F}(\mathfrak{D})$, consisting of all frames ϕ_x of $\mathfrak{D}$ such that the following two conditions hold

(1) each subspace $\phi_x\big(V^{(i)}\big)$ with $i < 0$ is an isotropic subspace of $\mathfrak{D}(x)$ w.r.t. the form ω_x;
(2) $\phi_x(V^{(-i-1)}) = \Big(\phi_x\big(V^{(i)}\big)\Big)^\angle$ for any $-1 - I \le i \le I$.

In the other words, the mapping $\phi_x \in \mathrm{F}_V(\mathfrak{D})$ induces a well defined, up to a multiplication on a nonzero constant, skew-symmetric bilinear form $\widetilde{\omega}_{\phi_x} = \phi_x^* \omega_x$ on V such that subspaces $V^{(i)}$ with $i < 0$ are isotropic and $V^{(-i-1)} = (V^{(i)})^\angle$ for any $i \in \{-I-1, \ldots, I\}$ w.r.t. $\widetilde{\omega}_{\phi_x}$. Besides, the form $\widetilde{\omega}_{\phi_x}$ induces the skew-symmetric bilinear form $\widetilde{\omega}_{\phi_x,\mathrm{gr}}$ on the graded space

$$\mathrm{gr}\, V = \bigoplus_{i=I_1+1}^{I} (V^{(i)}/V^{(i-1)}) \tag{4.2}$$

in the following way: assume that $\bar{x} \in V^{(j_1)}/V^{(j_1-1)}$ and $\bar{y} \in V^{(j_2)}/V^{(j_2-1)}$, then

(1) if $j_1 + j_2 = 0$, we put $\widetilde{\omega}_{\phi_x,\mathrm{gr}}(\bar{x}, \bar{y}) = \widetilde{\omega}_{\phi_x}(x, y)$, where x and y are representatives of $\bar{x}$ and $\bar{y}$ in $V^{(j_1)}(t)$ and $V^{(j_2)}(t)$ respectively;
(2) if $j_1 + j_2 = 0$, we put $\widetilde{\omega}_{\phi_x,\mathrm{gr}}(\bar{x}, \bar{y}) = 0$.

Let P be a fiber bundle over $\mathcal{M}$ endowed with the fixed fiberwise immersion $\mathfrak{F}$ to $\mathrm{F}_V(\mathfrak{D})$. One says that a fiber subbundle P of $\mathrm{F}_V(\mathfrak{D})$ has a *constant graded skew-symmetric form* if the forms $\widetilde{\omega}_{\mathfrak{F}(p),\mathrm{gr}}$ are the same, up to a multiplication on a nonzero constant, for all $p \in P$. For a rank 3 distribution D of maximal class and with the fixed diagram T as manifold $\mathcal{M}$ we take the manifold N, as a distribution $\mathfrak{D}$ we take the distribution Δ and as the bundle P we take the bundles $P_{k,l}$ for the corresponding k and l. Note that by our constructon the bundles $P_{k,l}$ have constant graded skew-symmetric form for any k and l.

Further, let $P_x = \pi^{-1}(x) \cap P$ be its fiber over x. Using the identifications above and the immersion $\mathfrak{F}$, one gets that the tangent space $T_p(P_{\pi(p)})$ to a fiber $P_{\pi(p)}$ at a point p can be identified with a subspace of $\mathfrak{gl}(V)$, which will be denoted by W_p. Indeed, to any vector A belonging to $T_p(P_{\pi(p)})$ we assign an element $\mathcal{I}_A$ of $\mathfrak{gl}(V)$ as follows: if $s \to p(s)$ is a smooth in

P such that $p(0) = p$ and $p'(0) = A$ then let $\mathcal{I}_A = \mathfrak{F}(p)^{-1} \circ \frac{d}{ds}\mathfrak{F}\big(p(s)\big)|_{s=0}$, where in the last formula by $\mathfrak{F}\big(p(s)\big)$ we mean the isomorphism between V and $\mathfrak{D}(\pi(p))$ corresponding to the frame $\mathfrak{F}\big(p(s)\big)$. Set $W_p = \{\mathcal{I}_A : A \in T_p(P_{\pi(p)})\}$. By analogy with the previous section, the filtration (4.1) induces a natural filtration on $\mathfrak{gl}(V)$ and, therefore, on any its subspace. The corresponding graded subspace $\operatorname{gr} W_p$ is called a *symbol of the bundle P at a point p*. Symbols are subspaces of $\operatorname{gr}\mathfrak{gl}(V)$, which in turn is naturally identified with $\mathfrak{gl}(\operatorname{gr}V)$. We say that the bundle P has *constant symbol* if its symbols at different points coincide.

In the sequel we shall deal only with fibre bundles P as above having constant graded skew-symmetric form and constant symbol. We denote this graded skew-symmetric form on $\operatorname{gr} V$ by $\mathfrak{w}$ and the symbol $\operatorname{gr} W_p$ by $\mathfrak{s}$. Note that by our construction the form $\mathfrak{w}$ is not identically zero.

Definition 3. *Let P be a fibre bundle endowed with the fixed fiberwise immersion $\mathfrak{F}$ to $F_V(\mathfrak{D})$ and with constant symbol $\mathfrak{s}$. Let W be a filtered linear space such that the dimensions of the spaces of its filtration are equal to the dimensions of the corresponding spaces of the filtration of W_p $(= T_p(P_{\pi(p)}))$. P is called a W-structure of frames on $\mathfrak{D}$, if the filtered tangent spaces W_p to fibers of P at different points are identified together with filtrations on them or , more precisely, if a smooth family $\{\psi_p\}_{p\in P}$ of isomorphisms $\psi_p : W \to W_p$, preserving the filtrations, is fixed.*

If the subbundle P is a reduction of the bundle $\mathrm{F}_V(\mathfrak{D})$ to a subgroup G of $GL(V)$ with the Lie algebra $\mathfrak{g}$, then P is automatically a $\mathfrak{g}$-structure: the filtration (4.1) induces a filtration on $\mathfrak{g}$ and the symbol of P is nothing but $\operatorname{gr}\mathfrak{g}$; the spaces $T_p(P_{\pi(p)})$ are naturally identified with $\mathfrak{g}$. This situation occurs for a rank 3 distribution with a rectangular diagram T. In this case the structure group is isomorphic to $ST(2,\mathbb{R}) \times GL(2,\mathbb{R})$. In the case of nonrectangular diagram the corresponding bundle $P_{k,l}$, $l > 0$, has constant symbol by Lemma 6. Moreover, as was shown in the previous section, on each fiber of $P_{k,l}$ there is a distinguished global frame, which is equivalent to the fact that the filtered tangent spaces W_p to fibers of $P_{k,l}$ at different points are identified (and also together with filtrations on them). So, $P_{k,l}$ is a W-structure as well.

4.2. **Prolongation procedure.** In the sequel for simplicity of presentation we suppose that $\pi\colon P \to \mathcal{M}$ is a W-structure, which is a fiber subbundle of $\mathrm{F}_V(\mathfrak{D})$. All constructions are generalized to arbitrary W-structures in an obvious way. We also assume that P has a constant graded skew-symmetric form on $\mathfrak{D}$. Let $\mathfrak{D}^{(1)} = \pi^*\mathfrak{D}$ be the pullback of $\mathfrak{D}$ by the projection π. One can define the partial soldering form on $\mathfrak{D}^{(1)}$, i.e., a field of linear maps $\theta_p : \mathfrak{D}^{(1)}(p) \mapsto V$ of V-valued partial one-form on $\mathfrak{D}^{(1)}$ given by:

$$\theta_p(X) = (p)^{-1}(d_p\pi(X)), \quad X \in \mathfrak{D}^{(1)}(p),$$

where, as before, a point $p \in P$ is identified with an isomorphism $p\colon V \to \mathfrak{D}\big(\pi(p)\big)$.

Consider a bundle Q over P with a fiber Q_p over a point p, consisting of all subspaces, which complete the spaces $W_p = \ker d_p\pi$ to $\mathfrak{D}^{(1)}(p)$. That is:

$$Q_p = \{H_p \subset \mathfrak{D}^{(1)}(p) \mid H_p \oplus W_p = \mathfrak{D}^{(1)}(p)\}.$$

Note that the partial soldering form θ defines an isomorphism of H_p with V for any horizontal subspace H_p. Fix a point $q \in Q_p$, $q = H_p$ and a pair of vectors v_1 and v_2 in V. Take two vector fields Y_1 and Y_2 in a neighbourhood U of p such that $\theta(Y_i) \equiv v_i$ in U and $Y_i(p) \in H_p$ for $i = 1, 2$. Set

$$\mathfrak{N}_q(v_1, v_2) = d_p\pi\left([Y_1, Y_2](p)\right). \tag{4.3}$$

It is clear that the vector $\mathfrak{N}_q(v_1, v_2) \in T_{\pi(p)}\mathcal{M}$ does not depend on a choice of a pair of vector fields Y_1 and Y_2 with the properties prescribed above.

Given a vector $v \in V$ define a vector $\operatorname{gr} v$ in the graded space $\operatorname{gr} V$ as follows: if $v \in V^{(i)}$, but $v \notin V^{(i-1)}$, then $\operatorname{gr} v$ is an equivalence class of v in $V^{(i)}/V^{(i-1)}$. By our constructions there exists two vectors $\bar v_1$ and $\bar v_2$ in V such that

$$\mathfrak{w}(\operatorname{gr} \bar v_1, \operatorname{gr} \bar v_2) \neq 0, \tag{4.4}$$

where, as before, $\mathfrak{w}$ is the graded skew-symmetric form on $\operatorname{gr} V$, associated with our bundle P. Condition (4.4) implies that

$$\widetilde{\omega}_p(\bar v_1, \bar v_2) \neq 0 \quad \text{for any } p, \tag{4.5}$$

where, as before, $\widetilde{\omega}_p(\bar v_1, \bar v_2) = d\alpha\big(p(\bar v_1), p(\bar v_2)\big)$ for some nonzero 1-form α, annihilating the distribution $\mathfrak{D}$. The condition (4.5) is in turn equivalent to the fact that the vector $\mathfrak{N}_q(\bar v_1, \bar v_2)$ is transversal to $\mathfrak{D}\big(\pi(p)\big)$.

Further, define a vector space $\widehat V = V \oplus \mathbb{R}\eta$ for some vector η. Then given a point $q \in Q_p$ one can define an extension of the isomorphism $p\colon V \mapsto \mathfrak{D}\big(\pi(p)\big)$ to an isomorphism $\chi_q\colon \widehat V \mapsto T_{\pi(p)}\mathcal{M}$ by setting $\chi_q(\eta) = \mathfrak{N}(q)(\bar v_1, \bar v_2)$. Finally, let $\operatorname{pr}\colon \widehat V \mapsto V$ be the canonical projection corresponding to the splitting $\widehat V = V \oplus \mathbb{R}\eta$. Then we can define the *structure function* $C\colon Q \to \operatorname{Hom}(\wedge^2 V, V)$, *associated with a pair of vectors* $\bar v_1$ *and* $\bar v_2$ as follows:

$$C(q)(v_1, v_2) = \operatorname{pr} \circ (\chi_q)^{-1}\big(\mathfrak{N}_q(v_1, v_2)\big). \tag{4.6}$$

Now take H_p and H_p' in Q_p. How $C(H_p)$ and $C(H_p')$ are related? First, recall that H_p and H_p' are subspaces in $\mathfrak{D}^{(1)}(p)$ complementary to the tangents space W_p to a fiber of P at a point p. Then we have a well-defined map:

$$\delta(H_p', H_p)\colon V \to W_p,$$

such that

$$X + \delta(H_p', H_p)(\theta(X)) \in H_p' \quad \text{for each } X \in H_p. \tag{4.7}$$

It is easy to see that the set of all subspaces at $p \in P$, which are complementary to W_p in $\mathfrak{D}(p)$, forms an affine space associated with a vector space $\operatorname{Hom}(V, W_p)$, i.e., Q is an affine bundle over P.

Second, recall that the *Spencer operator* S_p *of a pair* (V, W_p), where V is a vector space and W_p is a subspace of $\mathfrak{gl}(V)$ is defined as follows:

$$\begin{aligned} &S_p\colon \operatorname{Hom}(V, W_p) \to \operatorname{Hom}(\wedge^2 V, V), \quad \phi \mapsto S_p(\phi),\\ &S_p(\phi)\colon v_1 \wedge v_2 \mapsto \phi(v_1)v_2 - \phi(v_2)v_1, \quad v_1, v_2 \in V. \end{aligned} \tag{4.8}$$

Now assume, as before, that V is a vector space, $\bar v_1$ and $\bar v_2$ are vectors in V, $\widetilde{\omega}_p$ is a skew-symmetric form on V such that $\widetilde{\omega}_p(\bar v_1, \bar v_2) \neq 0$, and W_p is a subspace of $\mathfrak{gl}(V)$. To the quintuple $(V, \bar v_1, \bar v_2, \widetilde{\omega}_p, W_p)$ we assign a *modified Spencer operator* $\widetilde{S}_p$ in the following way:

$$\begin{aligned} &\widetilde{S}_p\colon \operatorname{Hom}(V, W_p) \to \operatorname{Hom}(\wedge^2 V, V), \quad \phi \mapsto \widetilde{S}_p(\phi),\\ &\widetilde{S}_p(\varphi)(v_1, v_2) = S_p(\varphi)(v_1, v_2) - \widetilde{\omega}_p(v_1, v_2)\frac{S_p(\varphi)(\bar v_1, \bar v_2)}{\widetilde{\omega}_p(\bar v_1, \bar v_2)}. \end{aligned} \tag{4.9}$$

Then by direct computation one can show that

$$C(H_p') = C(H_p) + \widetilde{S}_p\big(\delta(H_p', H_p)\big), \tag{4.10}$$

where $\delta(H'_p, H_p)$ is as in (4.7). The classical first prolongation $W_p^{(1)}$ of the subspace $W_p \subset \mathfrak{gl}(V)$ is defined to be the kernel of S_p. The modified first prolongation $W_p^{(1\mathrm{m})}$ of the subspace $W_p \subset \mathfrak{gl}(V)$ is defined to be the kernel of $\widetilde{S}_p$. In both cases $W_p^{(1)}$ and $W_p^{(1\mathrm{m})}$ are subspaces of $\mathfrak{gl}(V, W_p)$. It is clear that $W_p^{(1)} \subseteq W_p^{(1\mathrm{m})}$.

Note that the modified Spencer operator depends on a choice of a pair of vectors $\bar{v}_1$ and $\bar{v}_2$ in V, satisfying (4.4). The amazing thing is that *the modified first prolongation $W_p^{(1m)}$ does not depend on a choice of a pair of vectors $\bar{v}_1$ and $\bar{v}_2$ in V, satisfying* (4.5). This is because if one takes another pair of vectors $\tilde{v}_1$ and $\tilde{v}_2$ in V such that $\widetilde{\omega}_p(\tilde{v}_1, \tilde{v}_2) \neq 0$ and $\varphi \in W_p^{(1\mathrm{m})}$, then by (4.9)

$$\frac{S_p(\varphi)(\bar{v}_1, \bar{v}_2)}{\widetilde{\omega}_p(\bar{v}_1, \bar{v}_2)} = \frac{S_p(\varphi)(\tilde{v}_1, \tilde{v}_2)}{\widetilde{\omega}_p(\tilde{v}_1, \tilde{v}_2)}.$$

One can describe the modified first prolongation $W_p^{(1\mathrm{m})}$ in more symmetric way as follows:

$$W_p^{(1\mathrm{m})} = \{\varphi \in \mathrm{Hom}(V, W_p) \mid \widetilde{\omega}_p(\bar{v}_1, \bar{v}_2) S_p(\varphi)(v_1, v_2) - \widetilde{\omega}_p(v_1, v_2) S_p(\varphi)(\bar{v}_1, \bar{v}_2) = 0, \forall v_1, v_2, \bar{v}_1, \bar{v}_2 \in V\}$$

Besides, there is another convenient characterization of the modified first prolongation:

$$W_p^{(1\mathrm{m})} = \{\varphi \in \mathrm{Hom}(V, W_p) \mid \exists v \in V \text{ such that } S_p(\varphi)(v_1, v_2) = \widetilde{\omega}_p(v_1, v_2) v \ \forall v_1, v_2 \in V\} \tag{4.11}$$

More generally, as in the classical theory of prolongations, one can define the modified Spencer operator and the modified first prolongation also for a subspace $\mathfrak{W}$ of $\mathrm{Hom}(V, V_1)$, where V_1 is some linear space not necessary equal to V as before. In this case the Spencer and the modified Spencer operators are operators from $\mathrm{Hom}(V, \mathfrak{W})$ to $\mathrm{Hom}(V \wedge V, V_1)$ and they are defined by the same formulas, as in (4.8) and (4.9). The modified first prolongation is a kernel of the modified Spencer operator. There is also a description of the modified first prolongation analogous to (4.11), where we can consider W_p as a subspace of $\mathrm{Hom}(V, V_1)$ and take v from V_1. This slight generalization allows us to define inductively the modified higher order prolongations of W_p. Indeed, by construction $W_p^{(1\mathrm{m})} \subset \mathrm{Hom}(V, W_p)$. So, taking W_p as V_1, one can define

$$W_p^{(2\mathrm{m})} = \big(W_p^{(1\mathrm{m})}\big)^{(1\mathrm{m})}.$$

More generally, for any $i > 0$ we can consider $W_p^{(i\mathrm{m})}$ as a subspace in $\mathrm{Hom}(V, V_1)$ with $V_1 = W_p^{((i-1)\mathrm{m})}$ and set by induction $W_p^{((i+1)\mathrm{m})} = \big(W_p^{(i\mathrm{m})}\big)^{(1\mathrm{m})}$.

Lemma 7. *Suppose that W is a subalgebra in $\mathfrak{csp}(V)$ and $\dim W \geq 4$. Define a graded nilpotent Lie algebra $\mathfrak{g}_- = \mathfrak{g}_{-2} + \mathfrak{g}_{-1}$, where $\mathfrak{g}_{-2} = \mathbb{R}\eta$ for some element η, $\mathfrak{g}_{-1} = V$ and the Lie bracket given by $[v_1 + a_1\eta, v_2 + a_2\eta] = \widetilde{\omega}(v_1, v_2)\eta$. Set $\mathfrak{g}_0 = W$ and let $\mathfrak{g} = \sum_{i \geq -2} \mathfrak{g}_i$ be the Tanaka prolongation of the pair $(\mathfrak{g}_-, \mathfrak{g}_0)$. Then the p-th modified prolongation $W^{(pm)}$ of W coincides with the subspace $\mathfrak{g}_p$ in the Tanaka prolongation for any $p \geq 0$.*

Proof. According to [10] the subspace $\mathfrak{g}_{i+1}$ in the Tanaka prolongation is defined inductively as a set of all linear maps ϕ of degree $i+1$ from $\mathfrak{g}_-$ to $\sum_{j=-2}^{i} \mathfrak{g}_j$ such that

$$\phi([x_1, x_2]) = [\phi(x_1), x_2] + [x_1, \phi(x_2)], \quad \text{for all } x_1, x_2 \in \mathfrak{g}_-. \tag{4.12}$$

It is clear that any such map ϕ is uniquely defined by its restriction ψ to $\mathfrak{g}_{-1}$. Let $v = \phi(\eta) \in \mathfrak{g}_{i-1}$. Therefore, using induction, one can consider $\mathfrak{g}_{i+1}$ as a subspace in $\mathrm{Hom}(\mathfrak{g}_{-1}, \mathfrak{g}_i)$. Substituting

$x_1 = v_1$, $x_2 = v_2$, $v_1, v_2 \in V$ into equation (4.12) we get the following linear equation on a pair (ψ, v):

$$\widetilde{\omega}(v_1, v_2)v = [\psi(v_1), v_2] - [\psi(v_2), v_1], \quad \text{for all } v_1, v_2 \in V = \mathfrak{g}_{-1}. \tag{4.13}$$

This equation coincides with equation (4.11) on the modified prolongation $\mathfrak{g}_i^{(1\mathrm{m})}$ of $\mathfrak{g}_i$. It implies that $\mathfrak{g}_{i+1}$ is contained in $\mathfrak{g}_i^{(1\mathrm{m})}$.

In order to prove that these spaces are equal it is sufficient to show that if ϕ is a linear maps of degree $i+1$ from $\mathfrak{g}_-$ to $\sum_{j=-2}^{i} \mathfrak{g}_j$ such that(4.12) holds for any $x_1, x_2 \in \mathfrak{g}_{-1}(= V)$, then it holds for any $x_1, x_2 \in \mathfrak{g}_-$. It is trivial for $x_1 = x_2 = \eta$. So, it remains to consider only the case $x_1 = \eta$ and $x_2 = z \in V$. Since $[\eta, z] = 0$, in this case (4.12) is equivalent to

$$[\phi(\eta), z] + [\eta, \phi(z)] = 0. \tag{4.14}$$

Let us prove (4.14). Since $\dim V \geq 4$, we can choose x and y in V such that $[x, y] = \eta$ and $[x, z] = [y, z] = 0$. Then, using (4.12) for vectors from V, we get

$$[\phi(\eta), z] = \big[[\phi(x), y], z\big] + \big[[x, \phi(y)], z\big] = [\phi(x)y, z] - [\phi(y)x, z] \tag{4.15}$$

$$[\eta, \phi(z)] = \big[[x, \phi(z)], y]\big] + \big[x, [y, \phi(z)]\big] \tag{4.16}$$

Since $\phi(x) \in \mathfrak{g}_i$ and $[y, z] = 0$, we have $[\phi(x)y, z] = [\phi(x)z, y]$. Similarly, $[\phi(y)x, z] = [\phi(y)z, x]$. Substituting this to (4.15), we get

$$[\phi(\eta), z] = [\phi(x)z, y] - [\phi(y)z, x] \tag{4.17}$$

Further, since (4.12) holds on V and $[x, z] = 0$, then $[x, \phi(z)] = [z, \phi(x)]$. In the same way, $[y, \phi(z)] = [z, \phi(y)]$. Substituting it to (4.16) we get

$$[\eta, \phi(z)] = \big[[z, \phi(x)], y\big] + \big[x, [z, \phi(y)]\big] = -[\phi(x)z, y] + [\phi(y)z, x] \tag{4.18}$$

Identity (4.14) follows from (4.17) and (4.18). This completes the proof of the lemma. □

Remark 1. Note that the last lemma does not hold for $\dim V = 2$, as condition (4.11) becomes trivial in this case, and we get $W^{(1m)} = \mathrm{Hom}(V, W)$. In particular, if $W \neq 0$, then $W^{(im)}$ is also non-zero for any $i \geq 0$. On the other hand, Tanaka prolongation is non-trivial even for $\dim V = 2$. For example, if W consists of all diagonal matrices in a certain symplectic bases of V, then Tanaka prolongation of W becomes zero on the third step.

Corollary 1. *If W is a subalgebra in $\mathfrak{csp}(V)$ and $\dim V \geq 4$ then the sum $\mathbb{R}\eta + V + \sum_{i\geq 0} W^{(im)}$ has a natural structure of a graded Lie algebra, where $\deg \eta = -2$, $\deg V = -1$, and $\deg W^{(im)} = i$ for any $i \geq 0$.*

We can considered the graded analogs S_{gr} and $\widetilde{S}_{\mathrm{gr}}$ of Spencer operators S_p and $\widetilde{S}_p$. Namely, let S_{gr} be the Spencer operator of a pair $(\mathrm{gr}\, V, \mathfrak{s})$ and let $\widetilde{S}_{\mathrm{gr}}$ be the modified Spencer operator of the quintuple $(\mathrm{gr}\, V, \mathrm{gr}\, \bar{v}_1, \mathrm{gr}\, \bar{v}_2, \mathfrak{w}, \mathfrak{s})$, where, as before, $\mathfrak{s}$ is the symbol of our bundle P, i.e. $\mathrm{gr}\, W_p = \mathfrak{s}$ for any $p \in P$.

In general, operators S_p and S_{gr} or $\widetilde{S}_p$ and $\widetilde{S}_{\mathrm{gr}}$ are different and even operate on different spaces. However, they are closely related so that it is possible to carry prolongation procedure for the considered frame bundles on corank 1 filtered distribution in a similar way as for G-structures.

For any two filtered vector spaces V, W the spaces V^*, $\mathrm{Hom}(V, W)$, $S^k V$, $\wedge^k V$ are also naturally endowed with a filtration. In all these cases the associated graded vector spaces are naturally isomorphic to $(\mathrm{gr}\, V)^*$, $\mathrm{Hom}(\mathrm{gr}\, V, \mathrm{gr}\, W)$, $S^k(\mathrm{gr}\, V)$ and $\wedge^k(\mathrm{gr}\, V)$. Moreover, any subspace $U \subset V$ also inherits filtration together with a natural embedding of associated graded vector space $\mathrm{gr}\, U$ into $\mathrm{gr}\, V$. In the following we shall freely use these identifications.

Directly from definitions one can show that the Spencer operator S_p preserves the filtration. The same is true for the modified Spencer operator. Indeed, for this we actually have to show that if $v_1 \in V^{(j_1)}$, $v_2 \in V^{(j_2)}$, and the map $\varphi \in \big(\mathrm{Hom}(V, W_p)\big)^{(i)}$, then

$$\widetilde{\omega}_p(v_1, v_2) S_p(\varphi)(\bar{v}_1, \bar{v}_2) \in V^{(j_1+j_2+i)}. \tag{4.19}$$

For this assume that $\bar{v}_1 \in V^{(\bar{j}_1)}$ and $\bar{v}_2 \in V^{(\bar{j}_2)}$. Then from the fact that S_p preserves filtration it follows that $\widetilde{\omega}_p(v_1, v_2) S_p(\varphi)(\bar{v}_1, \bar{v}_2) \in V^{(\bar{j}_1+\bar{j}_2+i)}$. So, if $j_1 + j_2 \geq \bar{j}_1 + \bar{j}_2$, we are done. On the other hand, by definition of the form $\widetilde{w}_{p,\mathrm{gr}}$ it follows that $\bar{j}_1 + \bar{j}_2 = 0$. So, if $j_1 + j_2 < \bar{j}_1 + \bar{j}_2$, then $j_1 + j_2 < 0$, but then by condition (2) on filtration (**??**) one has $\widetilde{\omega}_p(v_1, v_2) = 0$. Hence (4.19) holds also in the case $j_1 + j_2 < \bar{j}_1 + \bar{j}_2$.

Besides, it is easy to show that both S_{gr} and $\widetilde{S}_{\mathrm{gr}}$ are the morphisms of degree 0 of graded spaces.

Lemma 8. *The operators S_{gr} and $\widetilde{S}_{\mathrm{gr}}$ coincides with the graded operators $\mathrm{gr}\, S_p$ and $\mathrm{gr}\, \widetilde{S}_p$ from $\mathrm{gr}\,\mathrm{Hom}(V, W_p)$ to $\mathrm{gr}\,\mathrm{Hom}(\wedge^2 V, V)$, associated with the operators S_p and $\widetilde{S}_p$ under natural identification of $\mathrm{gr}\,\mathrm{Hom}(V, W_p)$ with $\mathrm{Hom}(\mathrm{gr}\, V, \mathfrak{s})$ and $\mathrm{gr}\,\mathrm{Hom}(\wedge^2 V, V)$ with $\mathrm{Hom}(\wedge^2 \mathrm{gr}\, V, \mathrm{gr}\, V)$.*

Proof. For the classical Spencer operator the statement of the lemma follows directly from definitions. Let us check it for the modified Spencer operator. For this we have to check that if $\Xi_p\colon \mathrm{Hom}(V, W_p) \mapsto \mathrm{Hom}(V \wedge V, V)$ is defined by

$$\Xi_p(\varphi)(v_1, v_2) = \widetilde{\omega}_p(v_1, v_2) \frac{S_p(\varphi)(\bar{v}_1, \bar{v}_2)}{\widetilde{\omega}_p(\bar{v}_1, \bar{v}_2)},$$

then

$$\mathrm{gr}\,\Xi_p(\psi)(x_1, x_2) = \mathfrak{w}(x_1, x_2) \frac{S_{\mathrm{gr}}(\psi)(\mathrm{gr}\,\bar{v}_1, \mathrm{gr}\,\bar{v}_2)}{\mathfrak{w}(\mathrm{gr}\,\bar{v}_1, \mathrm{gr}\,\bar{v}_2)}. \tag{4.20}$$

Take, as before, $v_1 \in V^{(j_1)}$, $v_2 \in V^{(j_2)}$ and assume that $\bar{v}_1 \in V^{(\bar{j}_1)}$ and $\bar{v}_2 \in V^{(\bar{j}_2)}$. Then $\bar{j}_1 + \bar{j}_2 = 0$ and it is not difficult to show that

$$\begin{aligned} &\mathrm{gr}\,\Xi_p(\psi)(\mathrm{gr}\, v_1, \mathrm{gr}\, v_2) = 0, && \text{if } j_1 + j_2 > 0, \\ &\mathrm{gr}\,\Xi_p(\psi)(\mathrm{gr}\, v_1, \mathrm{gr}\, v_2) = \mathfrak{w}(\mathrm{gr}\, v_1, \mathrm{gr}\, v_2) \frac{S_{\mathrm{gr}}(\psi)(\mathrm{gr}\,\bar{v}_1, \mathrm{gr}\,\bar{v}_2)}{\mathfrak{w}(\mathrm{gr}\,\bar{v}_1, \mathrm{gr}\,\bar{v}_2)}, && \text{if } j_1 + j_2 \leq 0. \end{aligned}$$

Taking into account the definition of the graded skew-symmetric form $\mathfrak{w}$, we get that the last two relations are equivalent to (4.20). □

In the following we shall also need the *normalization conditions* for the prolongation procedure. These conditions are formally defined as any graded subspace $\mathrm{gr}\, N \subset \mathrm{Hom}(\wedge^2 \mathrm{gr}\, V, \mathrm{gr}\, V)$ such that:

$$\mathrm{Hom}(\wedge^2 \mathrm{gr}\, V, \mathrm{gr}\, V) = \mathrm{im}\,\widetilde{S}_{gr} \oplus \mathrm{gr}\, N.$$

Now let us prove the following general lemma:

Lemma 9. *Let $\Upsilon\colon A \to B$ be a mapping of arbitrary filtered vector spaces A, B preserving the filtration. Let $\mathrm{gr}\,\Upsilon\colon \mathrm{gr}\, A \to \mathrm{gr}\, B$ be the associated mapping of the corresponding graded vector spaces. Then we have:*

(1) $\mathrm{gr}(\ker \Upsilon) \subset \ker(\mathrm{gr}\,\Upsilon)$*;*

(2) *if C is any subspace in B such that*

$$\mathrm{gr}\, C \oplus \mathrm{im}\,\mathrm{gr}\,\Upsilon = \mathrm{gr}\, B, \tag{4.21}$$

then $C + \mathrm{im}\,\Upsilon = B$;

(3) *under the assumptions of the previous items, the space* $\mathrm{gr}\,\Upsilon^{-1}(C)$ *does not depend on* C *and coincides with* $\ker(\mathrm{gr}\,\Upsilon)$.

Proof.

1) Suppose that $a \in A^{(k)}$ and $\Upsilon(a) = 0$. Then $\mathrm{gr}\,\Upsilon(a + A^{(k-1)}) = \Upsilon(a) + B^{(k-1)} = 0$ and $a + A^{(k-1)} \in \mathrm{gr}\,A$ lies in the kernel of $\mathrm{gr}\,\Upsilon$.

2) Let b be any element in $B^{(k)}$. Then by assumption the element $b + B^{(k-1)} \in \mathrm{gr}\,B$ uniquely decomposes as $(c+C^{(k-1)})+(\Upsilon(a+A^{(k-1)}))$ for some elements $c+C^{(k-1)} \in \mathrm{gr}\,C$ and $a+A^{(k-1)} \in \mathrm{gr}\,A$. Hence, we see that $(b - c - \Upsilon(a))$ lies in $B^{(k-1)}$. Proceeding by induction we get that $b = c' + \Upsilon(a')$ for some elements $c' \in C$ and $a' \in A$.

3) Let $a \in \Upsilon^{-1}(C) \cap A^{(k)}$. Then $\mathrm{gr}\,\Upsilon(a + A^{(k-1)})$ lies in $\mathrm{gr}\,C$ and, hence, is equal to 0. Thus, we have $\mathrm{gr}\,\Upsilon^{-1}(C) \subset \ker(\mathrm{gr}\,\Upsilon)$.

To prove the opposite inclusion $\ker(\mathrm{gr}\,\Upsilon) \subset \mathrm{gr}\,\Upsilon^{-1}(C)$ we actually have to show that for any $a \in A^{(k)}$, satisfying $\Upsilon(a) \in B^{(k-1)}$, there exist $a' \in A^{(k)}$ such that $a - a' \in A^{(k-1)}$ and $\Upsilon(a') \in C$. For this let Υ_{k-1} be the restriction of Υ to $A^{(k-1)}$. Then from (4.21) it follows that $grC^{(k-1)} \oplus \mathrm{im}\,\mathrm{gr}\,\Upsilon_k = B^{(k-1)}$. Hence, by the previous item of the lemma we have

$$C^{(k-1)} + \mathrm{im}\,\Upsilon_{k-1} = B^{(k-1)}.$$

From this and the assumption that $\Upsilon(a) \in B^{(k-1)}$ it follows that there exist $c_{k-1} \in C^{(k-1)}$ and $a_{k-1} \in A^{(k-1)}$ such that $\Upsilon(a) = c_{k-1} + \Upsilon(a_{k-1})$. Therefore, as required a' one can take $a' = a - a_{k-1}$. Indeed, $a' - a = a_{k-1} \in A^{(k-1)}$ and $\Upsilon(a') = \Upsilon(a - a_{k-1}) = c_{k-1} \in C$. This completes the proof of the third item of the lemma. □

As a direct consequence of the two previous lemmas we get

Lemma 10. *Let*

$$\widetilde{S}_p\colon \mathrm{Hom}(V, W_p) \to \mathrm{Hom}(\wedge^2 V, V),$$
$$\widetilde{S}_{gr}\colon \mathrm{Hom}(\mathrm{gr}\,V, \mathfrak{s}) \to \mathrm{Hom}(\wedge^2 \mathrm{gr}\,V, \mathrm{gr}\,V)$$

be the modified Spencer operators associated with the quintuples $(V, \bar v_1, \bar v_2, \widetilde{\omega}_p, W_p)$ *and* $(\mathrm{gr}\,V, \mathrm{gr}\,\bar v_1, \mathrm{gr}\,\bar v_2, \mathfrak{w}, \mathfrak{s})$ *respectively.*

(1) *The subspace* $\mathrm{gr}\ker\widetilde{S}_p \subset \mathrm{Hom}(\mathrm{gr}\,V, \mathfrak{s})$ *associated with the subspace* $\ker\widetilde{S}_p \subset \mathrm{Hom}(V, W_p)$ *lies in* $\ker S_{gr}$. *In other words,* $\mathrm{gr}\,W_p^{(1m)}$ *is contained in* $\mathfrak{s}^{(1m)}$.

(2) *Let* $N \subset \mathrm{Hom}(\wedge^2 V, V)$ *be any subspace such that the associated graded space* $\mathrm{gr}\,N \subset \mathrm{Hom}(\wedge^2 \mathrm{gr}\,V, \mathrm{gr}\,V)$ *is complimentary to* $\mathrm{im}\,S_{gr}$. *Then* $N + \mathrm{im}\,S = \mathrm{Hom}(\wedge^2 V, V)$.

(3) *Let* N *be as in the previous item, and let*

$$W_{p,N}^{(1\mathrm{m})} = \{\phi \in \mathrm{Hom}(V, W_p) \mid \widetilde{S}_p(\phi) \in N\}.$$

Then the associated graded space $\mathrm{gr}\,W_{p,N}^{(1m)}$ *does not depend on* N *and coincides with* $\mathfrak{s}^{(1m)} = \ker\widetilde{S}_{gr}$.

Note that in general $\ker\widetilde{S}_p = W_p^{(1\mathrm{m})}$ has smaller dimension than $\ker\widetilde{S}_{gr} = \mathfrak{s}^{(1\mathrm{m})}$. But we shall need only the fact that $\ker\widetilde{S}_{\mathrm{gr}} = 0$ implies $\ker\widetilde{S}_p = 0$. Besides, although the space $\mathrm{gr}\,W_{p,N}^{(1\mathrm{m})}$ depends in general on a choice of a pair of vectors $(\bar v_1, \bar v_2)$ from V, satisfying (4.4), from item 3 of the previous lemma it follows that $\mathrm{gr}\,W_{p,N}^{(1\mathrm{m})}$ is independent not only of the subspace N but also of the choice of this pair.

Now everything is ready to describe the prolongation procedure for the W-structure P of frames with constant graded skew-symmetric form on $\mathfrak{D}$. Using filtrations on V and W, define a filtration on the space $V \oplus W$ as follows: First consider the following filtration:

$$V \oplus W = V^{(I)} \oplus W \supset V^{(I-1)} \oplus W \supset \ldots \supset V^{(-I-1)} \oplus W \supset \cdots \supset V^{(-I_1)} \oplus W = W$$

Second, take the filtration on W and make a shift of the indices (the weights) of their subspaces such that the index of W itself will be equal $-I_1$. The final filtration on $V \oplus W$ is obtained by pasting together these two filtration.

Fix a pair of vectors $(\bar{v}_1, \bar{v}_2)$ from V, satisfying (4.4). We call this pair the *initial pair for prolongation*. Fix an arbitrary subspace $N \subset \mathrm{Hom}(\wedge^2 V, V)$ such that the corresponding subspace $\mathrm{gr}\, N \subset \mathrm{Hom}(\wedge^2 \mathrm{gr}\, V, \mathrm{gr}\, V)$ is complimentary to $\mathrm{im}\, S_{gr}$, and, thus, satisfies the conditions of Lemma 10. The *first prolongation of P (subordinated to the subspace N and the initial pair $(\bar{v}_1, \bar{v}_2)$)* is a subbundle $P^{(1)}$ of Q consisting of all subspaces H_p complementary to W_p in $\mathfrak{D}^{(1)}(p)$ such that $C(H_p) \in N$. The bundle $P^{(1)}$ can be naturally identified with a subbundle of the frame bundle $\mathrm{F}_{V\oplus W}(\mathfrak{D}^{(1)})$. For this let $\{\psi_p\}_{p\in P}$ be a smooth family $\{\psi_p\}_{p\in P}$ of isomorphisms $\psi_p : W \to W_p$, as in Definition 3. Then for any point $p^{(1)} = (p, H_p) \in P^{(1)}$ one can define a linear map $L_{p^{(1)}} : V \oplus W \to T_p P$ such that $(L_{p^{(1)}})^{-1}|_{H_p} = p^{-1} \circ d_p \pi_{H_p}$ and $L_{p^{(1)}}|_W = \psi_p$, where in the first relation p is identified with an isomorphism from V to $\mathfrak{D}\big(\pi(p)\big)$ as in the beginning of the section. The skew-symmetric form on $V \oplus W$ corresponding to a point $p^{(1)}$ is such that its restriction to V coincides with the form $\tilde{\omega}_p$ and the subspace W belongs to its kernel. It implies that the bundle $P^{(1)}$ has a constant graded skew-symmetric form as well and that the initial pair of vectors $(\bar{v}_1, \bar{v}_2)$ satisfies the relation analogous to (4.4) for this graded skew-symmetric form. Furthermore, the tangent space to the fiber of $P^{(1)}$ can be identified with the subspace $W^{(1)}_{p,N}$ as in item (3) of Lemma 10. In particular, from the same item it follows that the first prolongation $P^{(1)}$ has constant symbol.

How to identify the tangent spaces to the fibers of $P^{(1)}$ at different points? Namely, we have to identify subspaces $W^{(1)}_{p_1,N}$ and $W^{(1)}_{p_2,N}$ for different $p_1, p_2 \in P$. For this let N' be any subspace of $\mathrm{Hom}(V, W)$ such that $\mathrm{gr}\, N'$ is complimentary to $\mathfrak{s}^{(1)}$. Then, since $\mathrm{gr}\, W^{(1)}_{p,N}$ is equal to $\mathfrak{s}^{(1)}$ for all $p \in P$, we see that N' is complimentary to $W^{(1)}_{p,N}$. Thus, by fixing an appropriate subspace N', we can identify all spaces $W^{(1)}_{p,N}$ with $W^1 = \mathrm{Hom}(V, W)/N'$. The space W^1 has the natural filtration, induced by the filtration on $\mathrm{Hom}(V, W)$. The identifications between $W^{(1)}_{p,N}$ and W^1 preserves the filtrations on this spaces.

As a conclusion, starting with W-structure P and fixing two spaces N and N' as above, one can define the first prolongation $P^{(1)}$ of P such that it is endowed with W^1-structure for an appropriate space W^1 and it has constant graded skew-symmetric form. In the same way, fixing appropriate spaces N_1 and N_1' one can define the second prolongation $P^{(2)} = (P^{(1)})^{(1)}$ and so on. The sequence $(N, N', N_1, N_1', \ldots)$ is called a *sequence of defining spaces for the prolongation procedure*.

Assume now that for some $k \in \mathbb{N}$ the kth modified prolongation $\mathfrak{s}^{(k\mathrm{m})}$ is equal to zero. It implies that choosing a sequence of defining subspaces we will get a canonical Ehresmann connection on the kth-prolongation $P^{(k)}$, subordinated to the chosen sequence of defining subspaces, and, consequently, the canonical frame on the corresponding corank 1 distribution $\mathfrak{D}^{(k)}$ of $P^{(k)}$. This frame can be extended to the canonical frame on $P^{(k)}$ by taking the Lie brackets of the vector fields in the frame, corresponding to the initial pair $(\bar{v}_1, \bar{v}_2)$. More precisely, a frame on $\mathfrak{D}^{(k)}$

is a family of isomorphisms $p^{(k)}\colon V\oplus W\oplus W^1\ldots\oplus W^{k-1}\mapsto \mathfrak{D}^{(k)}(p^{(k)})$. The additional vector field, which extends this frame to the frame on $P^{(k)}$, can be taken as $[p^{(k)}(v_1),p^{(k)}(v_2)]$, i.e. one can extend the map $p^{(k)}$ from the vector space $V\oplus W\oplus W^1\ldots\oplus W^{k-1}$ to the vector space $V\oplus W\oplus W^1\ldots\oplus W^{k-1}\oplus\mathbb{R}\nu$ such that $p^{(k)}(\nu)=[p^{(k)}(v_1),p^{(k)}(v_2)]$. Our constructions immediately imply the following

Theorem 2. *Assume that P is a W-structure with a constant graded skew-symmetric form and a constant symbol $\mathfrak{s}$. If the modified kth prolongation $\mathfrak{s}^{(km)}$ of $\mathfrak{s}$ vanishes for some $k\in N$, then with each such P, an initial pair of vectors for prolongation, and a sequence of defining spaces one can associate a canonical frame on a kth prolongation $P^{(k)}$ of P. Two such W-structures are locally equivalent if and only if their canonical frames corresponding to the same initial pair of vectors for prolongation and the same sequence of defining spaces are locally equivalent.*

The last theorem shows that in order to prove the existence of canonical frame on certain prolongation of the bundle P it is sufficient to analyze the modified prolongations of its symbol only.

5. Symbols and prolongations of W-structures for rank 3 distributions

Let, as above, V be a vector space endowed with a skew-symmetric bilinear form σ. Note that we do not require it to be non-degenerate, although most of the computations in this section will be done for non-degenerate form. Let, as above, W be any subspace in $\mathfrak{gl}(V)$. Again, in all computations below this will be actually a subalgebra in $\mathfrak{csp}(V)$.

Let us recall that according to (4.11) the modified prolongation of the subspace W is a subspace $W^{(1m)}$ in $\mathrm{Hom}(V,\mathfrak{g})$ consisting of all maps $\phi\colon V\to W$, for which we can find a vector $v\in V$ such that the following identity is satisfied:

$$\phi(v_1)v_2-\phi(v_2)v_1=\sigma(v_1,v_2)v,\quad \text{for all } v_1,v_2\in V. \tag{5.1}$$

Note that if such vector $v\in V$ exists for a given map ϕ, then it is unique. So, we can always identify $W^{(1m)}$ with a subspace in $\mathrm{Hom}(W,V)\times V$ complimentary to the second summand and consisting of all pairs (ϕ,v) satisfying (5.1).

It is clear that the standard first prolongation $W^{(1)}$ of W, defined as a set of all maps $\phi'\colon W\to V$ satisfying the equation:

$$\phi'(v_1)v_2-\phi'(v_2)v_1=0,\text{for all } v_1,v_2\in V,$$

is a subspace in the modified prolongation, as it corresponds to all pairs (ϕ,v) from the modified prolongation with $v=0$.

5.1. The symbol for non-rectangular diagrams.

Let us describe the symbol $\mathfrak{s}_{k,l}$ for the W-structure $P_{k,l}$ with $l>0$. Let $r=2k+l-1$ and let V be a vector space with a basis $\{e_1,\ldots,e_r,f_1,\ldots,f_r\}$.

Let us define linear maps $X,H,Y,Z_1,Z_2\in\mathrm{End}(V)$ as follows:

$$\begin{aligned}
&Xe_i=e_{i+1},\ Xf_i=f_{i+1},(i=1,\ldots,r-1),\quad Xe_r=Xf_r=0;\\
&He_i=(r+1-2i)e_i,Hf_i=(r+1-2i)f_i,(i=1,\ldots,r);\\
&Ye_i=(i-1)(r+1-i)e_{i-1},Yf_i=(i-1)(r+1-i)f_{i-1},(i=2,\ldots,r),\quad Ye_1=Yf_1=0;\\
&Z_1e_i=e_i;Z_1f_i=-f_i,(i=1,\ldots,r);\\
&Z_2e_i=e_i;Z_2f_i=f_i,(i=1,\ldots,r).
\end{aligned}$$

It is easy to check that the elements X, H, Y form a basis of a three-dimensional subalgebra in $\mathfrak{gl}(V)$ isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. Both elements Z_1 and Z_2 commute with this subalgebra. Denote by $\mathfrak{a}$ the 5-dimensional subalgebra in $\mathfrak{gl}(V)$ generated by these elements. Note also that the decomposition $V = V_e \oplus V_f$ is stable with respect to the action of $\mathfrak{a}$. Denote also by $\mathfrak{a}_0$ the subalgebra in $\mathfrak{a}$ of codimension 1 generated by elements H, Y, Z_1, Z_2.

According to Definition 2 define also a symplectic form σ on $V_e \oplus V_f$ by the formula

$$\begin{aligned} \sigma(e_i, e_j) &= \sigma(f_i, f_j) = 0; \\ \sigma(e_i, f_{r+1-i}) &= (-1)^i; \\ \sigma(e_i, f_j) &= 0,\ i + j \neq r + 1. \end{aligned}$$

We can see immediately that all five endomorphisms X, H, Y, Z_1, Z_2 preserve the symplectic form up to a scalar multiplier.

Consider now the action of $\mathfrak{sl}(2,\mathbb{R})$ (generated by X, H, Y) on the space of all linear maps $\mathrm{Hom}(V_f, V_e) \subset \mathfrak{gl}(V)$. The symplectic form σ gives us an isomorphism of $\mathfrak{sl}(2,R)$-modules V_f^* and V_e. Hence, the $\mathfrak{sl}(2,\mathbb{R})$-module $\mathrm{Hom}(V_f, V_e)$ is naturally isomorphic to $V_f^* \otimes V_e = V_e \otimes V_e$ and decomposes into irreducible $\mathfrak{sl}(2,\mathbb{R})$-submodules Π_{2r-2}, Π_{2r-4}, …, Π_2, Π_0 of dimensions $2r-1$, $2r-3$, …, 3, 1 respectively.

Under identification of V_f^* with V_e, the subspace of all elements from $\mathrm{Hom}(V_f, V_e)$ that preserve the symplectic form is naturally isomorphic to $S^2(V_e)$ and decomposes as $\mathfrak{sl}(2,\mathbb{R})$-module into the sum of submodules Π_{2r-2}, Π_{2r-6}, …, Π_2 (for even r) or Π_0 (for odd r).

Define the subspace $\mathfrak{p} \subset S^2(V_e) \subset \mathrm{Hom}(V_f, V_e) \subset \mathrm{End}(V)$ as the sum of irreducible $\mathfrak{sl}(2,\mathbb{R})$-submodules $\Pi_{2r-4k+2} = \Pi_{2l}$, Π_{2l-4}, …, Π_2 (or Π_0).

Theorem 3. *The symbol $\mathfrak{s}_{k,l}$, corresponding to the non-rectangular diagram of type (k,l), is equivalent to the subalgebra $\mathfrak{a} + \mathfrak{p} \subset \mathrm{End}(V)$, which in turn is equal to the algebra of infinitesimal symmetries of the corresponding flat curve $\mathfrak{F}_{k,l}$.*

Proof. To prove the theorem, we can consider any curve with the required diagram. We shall consider the flat curve $\mathfrak{F}_{k,l}$ as a simplest possible case and prove that in this case the W-structure is in fact a standard G-structure with a Lie algebra $\mathfrak{g} = \mathfrak{a} + \mathfrak{p}$.

Let us compute the symmetry group $\mathrm{Sym}(\mathfrak{F}_{k,l})$ of $\mathfrak{F}_{k,l}$ consisting of all linear transformations of V that preserve the form σ up to the constant and map the curve $\mathfrak{F}_{k,l}$ to itself. It is easy to see that this is a well-defined closed Lie subgroup in $GL(V)$. Denote by $\mathrm{Sym}_0(\mathfrak{F}_{k,l})$ the subgroup of $\mathrm{Sym}(\mathfrak{F}_{k,l})$ stabilizing the flag (1.2) and by $\mathrm{sym}(\mathfrak{F}_{k,l})$ and $\mathrm{sym}_0(\mathfrak{F}_{k,l})$ the corresponding subalgebras in $\mathfrak{gl}(V)$. Let $\mathfrak{g}_0$ be the subalgebra in $\mathfrak{gl}(V)$ consisting of all linear maps that stabilize the flag (1.2). Note, that, in particular, $\mathfrak{a}_0 + \mathfrak{p} \subset \mathfrak{g}_0$. It is easy to see that $\mathrm{sym}_0(\mathfrak{F}_{k,l})$ is a Lie algebra of the subgroup of $\mathrm{Sym}(\mathfrak{F}_{k,l})$ that stabilizes the point $\mathfrak{F}_{k,l}(0)$. As $\mathfrak{F}_{k,l}$ is one-dimensional, $\mathrm{sym}_0(\mathfrak{F}_{k,l})$ has codimension 1 in $\mathrm{sym}(\mathfrak{F}_{k,l})$, and, hence, $\mathrm{sym}(\mathfrak{F}_{k,l}) = \mathrm{sym}_0(\mathfrak{F}_{k,l}) + \mathbb{R}X$.

According to [2], the algebra $\mathrm{sym}_0(\mathfrak{F}_{k,l})$ can be defined as a largest subalgebra $\mathfrak{h}$ in $\mathfrak{g}_0$ that satisfies the condition $[X, \mathfrak{h}] \subset \mathfrak{h} + \mathbb{R}X$. We shall use this characterization to prove that in fact $\mathrm{sym}_0(\mathfrak{F}_{k,l}) = \mathfrak{a}_0 + \mathfrak{p}$. As $[X, \mathfrak{a}_0] \subset \mathfrak{a} = \mathfrak{a}_0 + \mathbb{R}X$ and $[X, \mathfrak{p}] \subset \mathfrak{p}$, we see that the subalgebra $\mathfrak{a}_0 + \mathfrak{p}$ does satisfy the above condition, and, thus, lies in $\mathrm{sym}_0(\mathfrak{F}_{k,l})$.

On the other hand, the dimension of $\mathrm{sym}_0(\mathfrak{F}_{k,l})$ can not exceed the dimension of the space of all (k,l)-quasisymplectic frames at the point $\mathfrak{F}_{k,l}(0)$. As follows from Proposition 1, the space of all (k,l)-quasisymplectic frames has dimension $1 + \sum_{r=0}^{[l/2]}(2l - 4r + 1)$. Taking into account the two-dimensional space of projective reparametrizations leaving a point $\mathfrak{F}_{k,l}(0)$ fixed, and a one-parameter group of all scalings of the symplectic form σ, we see that the space of all (k,l)-quasisymplectic frames has dimension $4+\sum_{r=0}^{[l/2]}(2l-4r+1)$. But we have $\dim\mathfrak{a}_0=4$ and $\dim\mathfrak{p}=\sum_{r=0}^{[l/2]}(2l-4r+1)$. This proves that $\mathrm{sym}_0(\mathfrak{F}_{k,l})=\mathfrak{a}_0+\mathfrak{p}$ and, thus, $\mathrm{sym}(\mathfrak{F}_{k,l})=\mathfrak{a}+\mathfrak{p}$.

Hence, the W-structure of all (k,l)-quasisymplectic frames associated with a flat curve is a standard G-structure with $G=\mathrm{Sym}_0(\mathfrak{F}_{k,l})$. In particular, any quasi-symplectic moving frame is in fact symplectic in case of the flat curve.

□

Let us prove now that this symbol is always of finite type and compute its prolongation explicitly in case $k=2$ (and $l>0$).

Lemma 11. *The modified prolongation $(\mathfrak{s}_{k,l})^{(1m)}$ coincides with the standard prolongation $\mathfrak{p}^{(1)}$ of the subspace $\mathfrak{p}$.*

Proof. Let (ϕ,v) be any element in $(\mathfrak{s}_{k,l})^{(1m)}$. Let us note that the space $\mathfrak{a}+\mathfrak{p}$ is in fact a subalgebra in $\mathfrak{gl}(V)$ preserving the subspace $V_e\subset V$. The restriction of this subalgebra to V_e is 4-dimensional and is generated by the elements X_{V_e}, H_{V_e}, Y_{V_e} and $(Z_1)_{V_e}$. In particular, it is equal to the image of the irreducible embedding of $\mathfrak{gl}(2,\mathbb{R})$ into $\mathfrak{gl}(V_e)$.

Let $(\phi,v)\in(\mathfrak{s}_{k,l})^{(1m)}$. Let us prove that $v=0$ and ϕ takes values in $\mathfrak{p}$. Consider the equation (5.1) in the following cases.

1°. $v_1,v_2\in V_e$. As the restriction of the symplectic form to V_e vanishes, we get $\phi(v_1)v_2=\phi(v_2)v_1$, which is exactly the equation for the standard prolongation of the restriction of $\mathfrak{a}_0+\mathfrak{p}$ to V_e. According to the result of Kobayashi–Nagano [7], the first prolongation of the irreducible embedding of $\mathfrak{gl}(2,\mathbb{R})$ is non-zero only if the dimension of the representation space does not exceed 3. In our case the lowest possible dimension of V_e is achieved when $k=2, l=1$ and is equal to 4. Thus, we see that $\phi(v_1)v_2=0$ for all $v_1,v_2\in V_e$. In particular, this implies that $\phi(V_e)$ lies in $\mathbb{R}(Z_1-Z_2)+\mathfrak{p}$.

2°. $v_1,v_2\in V_f$. As V_f is also an isotropic subspace, we also get $\phi(v_1)v_2=\phi(v_2)v_1$. Considering this equation modulo V_e, we again get that $\phi(v_1)v_2=0 \mod V_e$. This implies that $\phi(V_f)\subset\mathbb{R}(Z_1+Z_2)+\mathfrak{p}$.

Cases 1 and 2 above imply that ϕ can be decomposed as follows:

$$\phi(v_e+v_f)=\phi'(v_e+v_f)+\alpha(v_e)(Z_1-Z_2)/2+\beta(v_f)(Z_1+Z_2)/2,\quad \forall v_e\in V_e, v_f\in V_f, \tag{5.2}$$

where ϕ' takes values in $\mathfrak{p}$, $\alpha\in V_e^*$ and $\beta\in V_f^*$.

3°. $v_1=v_e\in V_e$, $v_2=v_f\in V_f$. Then we get:

$$\phi(v_e)v_f-\phi(v_f)v_e=\sigma(v_e,v_f)v.$$

Considering this equation modulo V_e and taking into account that $\mathfrak{p}$ vanishes on V_e and sends V_f to V_e, we get:

$$\alpha(v_e)v_f=\sigma(v_e,v_f)v \mod V_e,\quad \text{for any } v_e\in V_e, v_f\in V_f.$$

As the dimensions of V_e and V_f are at least 4, for any v_e we can find a non-zero vector $v_f\in V_f$ such that $\sigma(v_e,v_f)=0$. Hence, we see that $\alpha(v_e)=0$. In particular, we also see that $v\in V_e$.

Let us now fix an arbitrary $v_e\in V_e$. From (5.2) we have $\phi(v_e)=\phi'(v_e)$. Let us prove that $\phi'(v_e)=0$. Indeed, we have

$$\phi'(v_e)v_f=\sigma(v_e,v_f)v+\beta(v_f)v_e,\quad \text{for any } v_f\in V_f.$$

The element $\phi'(v_e)$ lies in $\mathfrak{p}$, while the right hand side defines a certain linear map from V_f to V_e, which has rank ≤ 2. However, as we shall see later (Corollary to Lemma 13), the space $\mathfrak{p}$ does not contain any non-zero elements of rank 2 or less. Thus, we get $\phi'(v_e)=0$ and v is proportional to v_e. As v_e can be arbitrary and the dimension of the subspace V_e is at least 4, this implies that $v = 0$ and $\beta = 0$.

Hence, ϕ takes values in $\mathfrak{p}$ and the modified prolongation $(\mathfrak{s}_{k,l})^{(1m)}$ coincides with $(\mathfrak{s}_{k,l})^{(1)}$. □

Lemma 12. *Let σ be any (possibly degenerate) skew-symmetric form on V. Suppose V is decomposed as $E \oplus F$, where both E and F are isotropic and F has a trivial intersection with kernel of σ. Let W be a subspace in* $\mathrm{Hom}(F,E) \subset \mathrm{End}(V)$ *not containing elements of rank* 1. *Then we have:*

1. *the modified prolongation $W^{(1m)}$ coincides with the standard prolongation $W^{(1)}$;*
2. *the space $W' = W^{(1m)} = W^{(1)}$, considered as a subspace in* $\mathrm{End}(V')$, *$V' = W \times V$, satisfies the conditions of the lemma for the decomposition $V' = E' \oplus F'$, where $E' = W \times E$, $F' = F$.*

Proof. Let $(\phi, v) \in W^{(1m)}$. Consider (5.1) for $v_e \in E$, $v_f \in F$. As $\phi(v_f)v_e = 0$, we get $\phi(v_e)v_f = \sigma(v_e, v_f)v$. So, the element $\phi(v_e)$ is a map of rank ≤ 1. Hence, by assumption of the lemma, we get $\phi(v_e) = 0$ for all $v_e \in V_e$. Since F has a trivial intersection with $\ker\sigma$, for any non-zero $v_f \in F$ there exists such $v_e \in E$ that $\sigma(v_e, v_f) \neq 0$. Hence, we also get $v = 0$. This proves that $W^{(1m)} = W^{(1)}$.

Let us prove the second part of the lemma. We have already proved above that $\phi(v_e) = 0$ for any $v_e \in E$. Hence, $W^{(1m)}$ lies in $\mathrm{Hom}(F', E') \subset \mathrm{End}(V')$. It is sufficient to show that W' does not have any elements of rank 1. Suppose, there is such an element. Then it can be represented as $v_f \mapsto \alpha(v_f)w_0$ for some non-zero $\alpha \in F^*$, $w_0 \in W$. Again, considering equation (5.1) for two vectors $v_1, v_2 \in F$, we get:

$$\alpha(v_1)w_0(v_2) = \alpha(v_2)w_0(v_1), \quad \text{for all } v_1, v_2 \in F.$$

We can always choose such $v_0 \in F$ that $\alpha(v_0) \neq 0$. Then we get $w_0(v) = \alpha(v)w_0(v_0)/\alpha(v_0)$ for any $v \in F$. Hence, w_0 has rank 1 as well, and this contradicts the assumption of the lemma. □

Corollary 2. *The modified prolongation $(\mathfrak{s}_{k,l})^{(im)}$ coincides with the standard prolongation $\mathfrak{p}^{(i)}$ of the subspace $\mathfrak{p}$ for any $i \geq 1$.*

Proof. Indeed, it is easy to see that subspace $\mathfrak{p} \subset \mathrm{Hom}(F,E) \subset \mathrm{End}(V)$ satisfies the condition of the lemma. Hence, $\mathfrak{p}^{(im)} = \mathfrak{p}^{(i)}$ for all $i \geq 0$. And according to Lemma 11, we have $(\mathfrak{s}_{k,l})^{(1m)} = \mathfrak{p}^{(1m)} = \mathfrak{p}^{(1)}$. Hence, $(\mathfrak{s}_{k,l})^{(im)} = \mathfrak{p}^{(im)} = \mathfrak{p}^{(i)}$ for any $i \geq 1$. □

To prove that the standard N-th prolongation $\mathfrak{p}^{(N)}$ vanishes for sufficiently large N, we shall accumulate the language of symplectic geometry and Poisson bracket. Let us introduce the space $\mathbb{R}^{2r}$ with coordinates $x_1, \dots, x_r, p_1, \dots, p_r$ and the symplectic from:

$$dx_1 \wedge dp_r - dx_2 \wedge dp_{r-1} + \dots + (-1)^r dx_r \wedge dp_1.$$

Note that the space of all polynomials $\mathbb{R}[x_i, p_j]$ with the Poisson bracket defined by this symplectic form is a Lie algebra with one-dimensional center generated by 1.

Let us identify $V_e \oplus V_f$ with subspace in $\mathbb{R}[x_i, p_j]$ consisting of all polynomials of degree 1 identifying basis vectors e_i with polynomials x_i and basis vectors f_i with p_i, $i = 1, \dots, r$. It is well-known that $\mathfrak{sp}(V_e \oplus V_f)$ can be identified then with the space of all quadratic polynomials in $\mathbb{R}[x_i, p_j]$, and k-th prolongation of $\mathfrak{sp}(V_e \oplus V_f)$ with the space of all polynomials of degree $k+2$.

Since the subspace $\mathfrak{p}$ lies in $S^2(V_e) = V_f^* \otimes V_e \subset \mathfrak{sp}(V_e \oplus V_f)$, we see that we can interpret it as a certain subspace of degree 2 polynomials in $x_1, \dots, x_r$. Let us define this subspace in a more geometric way using the language of algebraic geometry. Let $\mathbb{P}V_f$ be the projectivization of the subspace V_f and let C be the orbit of $\exp(\mathfrak{a})$ through the vector f_1. As $\mathfrak{a}_0$ and, hence $\exp(\mathfrak{a}_0)$, preserves $\mathbb{R}f_1$, we see that this orbit is 1-dimensional. It is evident that C is a closure of the orbit of $\exp(tX)$ and is a rational normal curve in $\mathbb{P}V_f$.

Let $\mathbb{P}^{r-1}$ be the projective space with homogeneous coordinates $[x_1 : x_2 : \ldots : x_r]$. Denote by C the normal rational curve in $\mathbb{P}^{r-1}$ given as an image of the Veronese embedding

$$\mathbb{P}^1 \to \mathbb{P}^{r-1}, \quad [s:t] \mapsto [s^{r-1} : s^{r-2}t : \ldots : t^{r-1}]. \tag{5.3}$$

By fixing a rational normal curve C in $\mathbb{P}^{r-1}$ we also fix the structure of $SL(2,\mathbb{R})$-module on $\mathbb{R}^r$.

Denote by $\mathcal{T}^nC$ the n-th tangential developable variety of C. Here we assume that $\mathcal{T}^0C = C$. If $\mathcal{V}$ is any algebraic variety in $\mathbb{P}^{r-1}$, we denote, as usual, by $I(\mathcal{V})$ the ideal of homogeneous polynomials in $x_1,\ldots,x_r$ vanishing on $\mathcal{V}$. We shall also denote by $I_n(\mathcal{V})$ the subspace of all polynomials of degree n in $I(\mathcal{V})$. Denote also by $\mathcal{S}_n\mathcal{V}$ the n-th secant variety of $\mathcal{V}$, which is defined as an algebraic closure of the union of $(n-1)$-planes in $\mathbb{P}^{r-1}$ passing through n points from $\mathcal{V}$. By definition we set $\mathcal{S}_1(\mathcal{V}) = \mathcal{V}$.

Let us recall a standard result from algebraic geometry:

Lemma 13 ([5]). *Let*

$$S^2(\mathbb{R}^{r,*}) = \sum_{i\geq 0} \Pi_{2r-4i-2}$$

be the decomposition of the $SL(2,\mathbb{R})$-module $S^2(\mathbb{R}^{r,*})$ into the sum of the irreducible submodules, where Π_m is a unique submodule of dimension $m+1$.

Then the space $I_2(\mathcal{T}^sC)$ of all degree 2 polynomials vanishing on $\mathcal{T}^sC$ is equal to $\sum_{i\geq s} \Pi_{2r-4i-2}$.

Corollary 3. *The space $I_2(C)$ does not contain non-zero elements of rank ≤ 2.*

Proof. According to the lemma, any element from $I_2(C)$ should vanish on C. Suppose that $I(C)$ contains any element F of rank ≤ 2. Then it lies in a linear combination of polynomials G^2, GH, H^2 for certain homogeneous degree 1 polynomials G, H. We can always extend our base field to $\mathbb{C}$ and decompose F into the product of two linear polynomials G' and H'. As F vanishes on the rational curve, we see that either G' or H' should also vanish on C. However, this is impossible as C does no lie in any proper linear subspace in $\mathbb{P}^{r-1}$. □

Using Lemma 13 above we get

Theorem 4. *The subspace $\mathfrak{p} \subset S^2(V_e)$ can be identified with the space of all quadratic polynomials in $x_1,\ldots,x_r$ that vanish at $(k-2)$-th tangent developable variety of C.*

The n-th prolongation of $\mathfrak{p}$ in contained in $I_{n+2}(\mathcal{V})$, where $\mathcal{V} = \mathcal{S}_{n+1}(\mathcal{T}^{k-2}C)$ is the $(n+1)$-th secant variety of $(k-2)$-th tangential variety to the rational normal curve C.

Proof. The first part of the theorem immediately follows from definition of $\mathfrak{p}$ and Lemma 13.

To prove the second part, let us consider a more general case. Let $\mathcal{V}$ be an arbitrary algebraic variety in $\mathbb{P}^{r-1}$ and let $I_2(\mathcal{V})$ be the set of all quadratic polynomials in $x_1,\ldots,x_r$ vanishing at $\mathcal{V}$. If we consider $I_2(\mathcal{V})$ as a subspace in $S^2(V_e) \subset \mathrm{Hom}(V_f, V_e) \subset \mathfrak{gl}(V)$, then its n-th prolongation can be identified with polynomials $F(x_1,\ldots,x_r)$ of degree $n+2$ such that $\frac{\partial^n F}{\partial x_1^{\alpha_1}\ldots\partial x_r^{\alpha_r}} \in I_2(\mathcal{V})$ for all multi-indices $\alpha = (\alpha_1,\ldots,\alpha_r)$ with $|\alpha| = n$. It is clear that any such polynomial and all its partial derivatives of degree $\leq n$ lie in $I(\mathcal{V})$.

Let us prove that any such polynomial F vanishes identically at $\mathcal{S}_{n+1}(\mathcal{V})$. It is sufficient to prove that it vanishes at any secant n-plane of $\mathcal{V}$. Indeed, let $p_0,\ldots,p_n \in \mathcal{V}$ be the set of $n+1$ points and let

$$W\colon \mathbb{P}^n \to \mathbb{P}^{r-1}, \quad [y_0 : y_1 : \cdots : y_n] \mapsto y_0p_0 + y_1p_1 + \ldots y_np_n;$$

be the embedding of the corresponding secant n-plane. Then $F \circ W$ is a polynomial of degree $n+2$ on $\mathbb{P}^n$ that vanishes at basis points $q_i \in \mathbb{P}^n$, $i = 0, \dots, n$, and, in addition, all its derivatives of degree $\leq n$ also satisfy this property. This immediately implies that $F \circ W = 0$. Hence, F vanishes identically at $\mathcal{S}_{n+1}(\mathcal{V})$. □

Corollary 4. *The subspace $\mathfrak{p}$, and, hence, the symbol $\mathfrak{s}_{k,l}$ is of finite type.*

Proof. Since the rational normal curve C is non-degenerate, its r-th secant variety coincides with $\mathbb{P}^{r-1}$. Hence, $\mathcal{S}_{n+1}(\mathcal{T}^{k-2}C) = \mathbb{P}^{r-1}$ for any $n \geq r-1$. Then the ideal $I(\mathcal{V})$ is trivial, and $I_{n+2}(\mathcal{V}) = \{0\}$. □

Let us consider the case $k = 2$ in more detail. We have a well-known description of n-secant varieties of the rational normal curve (5.3):

Lemma 14 ([6])**.** *Let $\mathcal{V} = \mathcal{S}_n(C)$ be the n-secant variety of C, where $n \geq 0$. Then the ideal $I(\mathcal{V})$ is generated (as an ideal) by $I_{n+2}(\mathcal{V})$. The space $I_{n+2}(\mathcal{V})$, in its turn, is generated (as a linear space) by all rank $n+2$ minors of the matrix:*

$$\begin{pmatrix} x_1 & x_2 & x_3 & \dots & x_{r-\alpha} \\ x_2 & x_3 & x_4 & \dots & x_{r-\alpha+1} \\ \vdots & & & & \\ x_{\alpha+1} & x_{\alpha+2} & x_{\alpha+2} & \dots & x_r \end{pmatrix},$$

where α is an arbitrary integer between n and $r/2$.

In particular, we see that in case $k = 2$ the space $\mathfrak{p}^{(n)}$, $n \geq 0$ coincides with the ideal $I_{n+2}(\mathcal{S}_n C)$ and is described explicitly by Lemma 14.

5.2. **The symbol in case of rectangular diagrams.** In this subsection we describe the symbol $\mathfrak{s}_{k,0}$ for W-structure $P_{k,0}$, constructed in case of rectangular diagrams. Let $r = 2k-1$. We assume that $k \geq 3$, so that $r \geq 5$. The case $k = 2$ corresponds to non-degenerate $(3,6)$ distributions, and has been studied by Bryant [1].

Let U be an irreducible $\mathfrak{sl}(2,\mathbb{R})$-module of dimension r. Define a Lie algebra $\mathfrak{a}$ as a direct product $\mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{gl}(2,\mathbb{R})$ and define a natural action of $\mathfrak{a}$ on the tensor product $V = U \otimes \mathbb{R}^2$. We shall identify $\mathfrak{a}$ with the subalgebra in $\mathfrak{gl}(V)$ defined by the action of $\mathfrak{a}$ on V.

There is a unique (up to a constant) $(\mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{sl}(2,\mathbb{R}))$-invariant symplectic form σ on V, which can be defined as a product of a (unique up to a constant) $\mathfrak{sl}(2,\mathbb{R})$-invariant non-degenerate symmetric form on U and the standard skew-symmetric form on $\mathbb{R}^2$ preserved by the second $\mathfrak{sl}(2,\mathbb{R})$.

Theorem 5. *The symbol $\mathfrak{s}_{k,0}$ for rectangular diagrams is equivalent to the subalgebra $\mathfrak{a} \subset \mathfrak{gl}(V)$, which in turn is equal to the algebra of infinitesimal symmetries of the flat curve $\mathfrak{F}_{k,0}$.*

The first modified prolongation $(\mathfrak{s}_{k,0})^{(1m)}$ is trivial for $k \geq 2$.

Proof. The first part of the theorem can be proved in the same way as Theorem 3.

Let us prove the second part. Denote by $\{e_1, e_2, \dots, e_r\}$ the standard basis of the $\mathfrak{sl}(2,\mathbb{R})$-module U and decompose V as $\oplus_{i=1}^r V_i$, where $V_i = \mathbb{R}e_i \otimes \mathbb{R}^2$. Then we have $\mathfrak{a}.V_i \subset V_{i-1} + V_i + V_{i+1}$ for $i = 2, 3, r-1$ and $\mathfrak{a}.V_1 \subset V_1 + V_2$, $\mathfrak{a}.V_r \subset V_{r-1} + V_r$. Note also that $(e_i, e_j) = 0$ for any $i + j \neq 6$. In particular, we get $\sigma(V_i, V_j) = 0$ for any $i + j \neq 6$.

Let (ϕ, v) be the element of the modified prolongation of $\mathfrak{a}$. Consider equation (5.1) for $v_1 \in V_1$ and $v_2 \in V_{r-1}$. As $\sigma(v_1, v_2) = 0$, we get $\phi(v_1)v_2 = \phi(v_2)v_1$. Since $\phi(v_1)v_2$ lies in $\mathfrak{a}.V_{r-1} \subset V_{r-2} + V_{r-1} + V_r$ and $\phi(v_2)v_1$ lies in $V_1 + V_2$, we see that both parts of this equality should vanish. This is possible only if $\phi(v_1)$ lies in $\mathbb{R}H + \mathfrak{gl}(2,\mathbb{R})$, and, in particular $\phi(V_1)V_i \subset V_i$ for all $i = 1,\dots,r$. Similarly, we can prove that $\phi(V_r)V_i \subset V_i$ for $i = 1,\dots,r$.

Consider now equation (5.1) for $v_1 \in V_1$ and $v_2 \in V_r$. We get $v \in \phi(V_1).V_r + \phi(V_r).V_1 \subset V_1 + V_r$. On the other hand, if we consider $v_1, v_2 \in V_k$ (recall that $r = 2k-1$), we see that $v \in \mathfrak{a}.V_k \subset V_{k-1} + V_k + V_{k+1}$. As $k \geq 3$, we get $v = 0$ and the modified prolongation $\mathfrak{a}^{(1m)}$ coincides with the standard prolongation $\mathfrak{a}^{(1)}$. But according to Kobayashi–Nagano [7], $\mathfrak{a}^{(1)}$ is trivial for $r \geq 5$. □

Finally note that the Main Theorem, formulated in the Introduction, is obtained by combining Theorem 2, Lemma 7, Theorem 3, Corollary 4, and Theorem 5.

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