But for some classes of matrices a basis of eigenvectors always exists, for example **symmetric matrices**: $A^T = A \iff a_{ij} = a_{ji}$ for all i,j

-1-

Example (the case of a repeated root, when we still have a basis of eigenvectors)

Find the general solution of the system

$$x_1' = 3x_1 + 2x_2 + 4x_3$$

$$x_2' = 2x_1 + 2x_3$$

$$x_3' = 4x_1 + 2x_2 + 3x_2$$

Matrix form

$$x' = \underbrace{\begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}}_{A} x$$

$\boxed{A \text{ is symmetric matrix} \Rightarrow \text{diagonizable}}$

1) Characteristic equation

$$\det(A - \lambda I) = \det\begin{pmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{pmatrix} = (3-\lambda)((\lambda-3)\lambda - 4) -$$

$$-2(2(3-\lambda)-8) + 4(4+4\lambda) = (3-\lambda)\underbrace{(\lambda^2 - 3\lambda - 4)}_{(\lambda+1)(\lambda-4)} -$$

$$-4(\underbrace{3-\lambda-4}_{-\lambda-1}) + 16(\lambda+1) =$$

$$= (\lambda+1)\underbrace{(-(\lambda-3)(\lambda-4) + 20)}_{(-\lambda^2+7\lambda+8)} = -(\lambda+1)\underbrace{(\lambda^2 - 7\lambda - 8)}_{(\lambda+1)(\lambda-8)} =$$

$$= -(\lambda+1)^2(\lambda-8) = 0$$

Only 2 eigenvalues

$\lambda = -1$ is zero of multiplicity 2 of the char. polynomial

$\lambda = 8$ is zero of multiplicity 1 of the char

$\Rightarrow \lambda = -1$ has algebraic multiplicity 2

2) Eigenspace of $\lambda = -1$:

$$(A+I)\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}}_{\text{3 equations are dependent} \Leftrightarrow \text{1 equation}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\underbrace{2v_1 + v_2 + 2v_3 = 0}_{\text{equation of a plane}}$$

(~~remember calculus 3?~~ The set of all eigenvectors of -1 and 0 vector is a plane.)

Choose a basis in this plane

$\Downarrow$

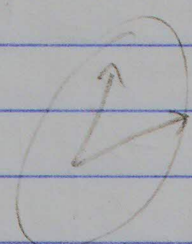
geom. multiplicity of $\lambda = -1$ is also 2

for example take $v_2 = 1$, $v_3 = 0 \Rightarrow v_1 = -\frac{1}{2}$

$v_2 = 0$, $v_3 = 1 \Rightarrow v_1 = -1$

$$\underbrace{\begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix}}_{\text{or } \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}} \text{ and } \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

(i.e. geom. multiplicity = alg. multiplicity)



3) Eigenvector of $\lambda = 8$

$$(A-8I)\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{array}{c} 4\times \; 2\times \\ \times 5 \\ 5 \end{array}\begin{pmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{pmatrix} = \begin{pmatrix} -5 & 2 & 4 \\ 0 & -36 & 18 \\ 0 & 18 & -9 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} =$$

$$\Leftrightarrow \begin{array}{l} -5v_1 + 2v_2 + 4v_3 = 0 \\ -2v_2 + v_3 = 0 \Rightarrow v_3 = 2v_2 \Rightarrow -5v_1 + 2v_2 + 8v_2 = 0 \Rightarrow v_1 = 2v_2 \end{array}$$

Take $v_2 = 1 \Rightarrow v_3 = 2,\ v_1 = 2 \Rightarrow \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$

$$c_1 e^{-t} \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + e^{8t} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$