Repeated roots (Example for $n=3$)

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More general point of view on repeated roots
E_λ is the set of all eigenvectors of λ and a 0 vector

To an eigenvalue one can assign 2 integers:

E_λ is a vector

1) algebraic multiplicity $\to$ the multiplicity of λ in the char. polynomial $\det(A-\lambda I)$
2) geometric multiplicity

dimension of $E_\lambda \to$ geometric multiplicity

Prop geometric multiplicity $\leq$ algebraic multiplicity

Thm A basis of eigenvectors exists $\iff$ for any eigenvector the geometric multiplicity $=$ algebraic multiplicity

(The case when for some eigenvalue geom mult $<$ alg. mult.)

$n=3$

$$\begin{cases} x_1' = -3x_1 + 3x_2 + 8x_3 \\ x_2' = 11x_1 - 4x_2 - 17x_3 \\ x_3' = -5x_1 + 3x_2 + 10x_3 \end{cases}$$

Find the general solution

Solution

$$A = \begin{pmatrix} -3 & 3 & 8 \\ 11 & -4 & -17 \\ -5 & 3 & 10 \end{pmatrix}$$

Find the eigenvalues:

$$\det(A-\lambda I) = \begin{vmatrix} -3-\lambda & 3 & 8 \\ 11 & -4-\lambda & -17 \\ -5 & 3 & 10-\lambda \end{vmatrix} =$$

$$= -(\lambda+3)\left((\lambda-10)(\lambda+4)+51\right) - 3\left(11(10-\lambda) - 85\right) +$$

$$+ 8\left(33 - 5(\lambda+4)\right) =$$

$$= -(\lambda+3)(\lambda^2-6\lambda+11) - 3(25-11\lambda) +$$

$$+8(13-5\lambda) = -\lambda^3 - 3\lambda^2 + 6\lambda^2 + 18\lambda - 11\lambda - 33 -$$

$$-75 + 33\lambda + 104 - 40\lambda =$$

$= -\lambda^3 + 3\lambda^2 - 4 = 0$. Then $\lambda = -1$ is a root $\Rightarrow$

$\lambda^3 - 3\lambda^2 + 4$ is divisible by $\lambda+1$. Make the long division

$$\lambda^3 - 3\lambda^2 + 4 = 0$$

$$\begin{array}{r|l} & \lambda^2 - 4\lambda + 4 \\ \hline \lambda+1 & \lambda^3 - 3\lambda^2 + 4 \\ & \lambda^3 + \lambda^2 \\ \hline & -4\lambda^2 + 4 \\ & -4\lambda^2 - 4\lambda \\ \hline & 4\lambda + 4 \end{array}$$

$\Rightarrow \lambda^3 - 3\lambda^2 + 4 = 0 \Leftrightarrow$

$(\lambda-2)^2(\lambda+1) = 0 \Rightarrow \lambda = 2 \qquad \lambda = -1$

algebraic mul

$\lambda = 2$ - algebraic multiplicity 2

- Find the space of eigenvectors for $\lambda = 2$

$(A-2I)\,v = 0$

$$\left(\begin{array}{ccc|c} -5 & 3 & 8 & 0 \\ 11 & -6 & -17 & 0 \\ -5 & 3 & 8 & 0 \end{array}\right) \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to 5R_2 + 11R_1} \begin{array}{ccc} -5 & 3 & 8 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{array}$$

$-5v_1 + 3v_2 + 8v_3 = 0$

$3v_2 + 3v_3 = 0 \quad \therefore v_2 = -v_3$

$v_3 = 1, \; v_2 = -1$

$-5v_1 - 3 + 8 = 0 \Rightarrow v_1 = 1$

· The eigenspace E_2 is 1-dim and generated by $v = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ $\Rightarrow$

geom. multiplicity is equal to 1

geom. mult < algebraic mult.

(i.e. in this case geom. multiplicity < alg. multiplicity.

· Generalized eigenvector: Solve

$(A-2I)w = v$.

$$\left(\begin{array}{ccc|c} -5 & 3 & 8 & 1 \\ 11 & -6 & -17 & -1 \\ \cancel{-5} & \cancel{3} & \cancel{8} & 1 \end{array}\right) \overset{R_2 \to 5R_2 + 11R_1}{\sim} \left(\begin{array}{ccc|c} -5 & 3 & 8 & 1 \\ 0 & 3 & 3 & +6 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim$$

The third row coincides with the first row

$$\left(\begin{array}{ccc|c} -5 & 3 & 8 & 1 \\ 0 & 1 & 1 & 2 \end{array}\right)$$

$-5w_1 + 3w_2 + 8w_3 = 1$
$w_2 + w_3 = 2$

$w_3 = 1 \Rightarrow v_2 = 1 \Leftrightarrow$

$-5v_1 + 3 + 8 = 1 \Rightarrow -5v_1 = -10 \Leftrightarrow v_1 = 2$

$w = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

Another way:

$(A-2I)^2 = \begin{pmatrix} 18 & -9 & -27 \\ -36 & 18 & 54 \\ 18 & -9 & -27 \end{pmatrix}$

$\downarrow$

Gen eigenvectors of order 2 - 2 dim space

$\underbrace{2v_1 - v_2 - 3v_3 = 0}$

$v = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \to$ Take as w something which is not collinear to v

- Eigenspace of $\lambda = -1$

$$Z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$$

$$(A+I)\,Z = \begin{pmatrix} -2 & 3 & 8 \\ 11 & -3 & -17 \\ -5 & 3 & 11 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} -2 & 3 & 8 & 0 \\ 11 & -3 & -17 & 0 \\ -5 & 3 & 11 & 0 \end{array}\right) \xrightarrow[R_3 \to 2R_3 - 5R_1]{R_2 \to 2R_2 + 11R_1}$$

Lo1

$$\to \left(\begin{array}{ccc|c} -2 & 3 & 8 & 0 \\ 0 & 27 & 54 & 0 \\ 0 & -9 & -18 & 0 \end{array}\right) \xrightarrow[R_3 \to 3R_3 + R_2]{}$$

$$\to \left(\begin{array}{ccc|c} -2 & 3 & 8 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$-2z_1 + 3z_2 + 8z_3 = 0$

$z_2 + 2z_3 = 0$

$z_3 = -1 \quad z_2 = 2, \quad -2z_1 + 6 - 8 = 0$

$-2z_1 = 2 \Rightarrow z_1 = -1$

$$\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} v & w & z \end{pmatrix}}_{Q}$$

$Av = 2v$

$Aw = v + 2w$

$Az = -z$

$$B = \begin{matrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{matrix}$$

$\begin{pmatrix} v & w & z \end{pmatrix} e^{Bt}$ is the fund matrix

$$e^{Bt} = \begin{pmatrix} e^{2t} & te^{2t} & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{-t} \end{pmatrix} \Rightarrow$$

$\begin{pmatrix} e^{2t}v & te^{2t}v + e^{2t}w & e^{-t}z \end{pmatrix}$ is a fundamental matrix $\Rightarrow$

$$x(t) = c_1 e^{2t}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + c_2\left(te^{2t}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + e^{2t}\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}\right) + c_3 e^{-t}\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$$ is

the general solution