A CONGRUENCE MODULO FOUR FOR REAL SCHUBERT CALCULUS WITH ISOTROPIC FLAGS

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Abstract. We previously obtained a congruence modulo four for the number of real solutions to many Schubert problems on a square Grassmannian given by osculating flags. Here, we consider Schubert problems given by more general isotropic flags, and prove this congruence modulo four for the largest class of Schubert problems that could be expected to exhibit this congruence.

Introduction

The number of real solutions to a system of real equations is congruent to the number of complex solutions modulo two. In [8], we established a congruence modulo four for many symmetric Schubert problems given by osculating flags, leaving as a conjecture a stronger form of that result. We prove this conjecture for symmetric Schubert problems given by flags that are isotropic with respect to a symplectic form, giving a simpler proof of a stronger and more basic result than that obtained in [8].

This congruence modulo four follows from a result on the real points in fibers of a map between real varieties equipped with an involution. When the fixed point set of the involution has codimension at least two, the number of real points satisfies a congruence modulo four. There is an involution acting on symmetric Schubert problems given by isotropic flags and we can compute the dimension of the fixed point locus in a universal family of Schubert problems. Our inability to compute this dimension when the flags are osculating was the obstruction to establishing the conjecture in [8].

The congruence modulo four often implies a non-trivial lower bound on the number of real solutions to a symmetric Schubert problem given by isotropic flags. Similar lower bounds and congruences in real algebraic geometry have been of significant interest [1, 3, 6, 9, 10, 11, 17, 18, 20, 24]. Another topological study was recently made of this phenomenon in the Schubert calculus [5], and delicate lower bounds [14] were given by computing the signature of a hermitian matrix arising in the proof of the Shapiro Conjecture [15].

In Section 1 we state our main result, whose proof occupies Section 2.

1. Symmetric Schubert Problems

Let V be a complex vector space of dimension $2m$ equipped with a nondegenerate alternating form $\langle\ ,\ \rangle\colon V\otimes V\to\mathbb{C}$. Write $\overline{W}$ for the complex conjugate of a point, vector,

2010 *Mathematics Subject Classification.* 14N15, 14P99.

Key words and phrases. Lagrangian Grassmannian, Wronski map, Shapiro Conjecture.

Research of Sottile and Hein supported in part by NSF grant DMS-1001615.

subspace, or variety W. A variety W is *real* if it is defined by real equations; equivalently, if $\overline{W}=W$. Write $W(\mathbb{R})$ for the real points of a real variety W, those that are fixed by complex conjugation. Write S_a for the symmetric group of permutations of $\{1,\ldots,a\}$.

The set of m-dimensional linear subspaces of V forms the Grassmannian, $\mathrm{Gr}(m,V)$, which is a manifold of dimension m^2. A *flag* is a sequence $F_\bullet\colon F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_{2m}=V$ of linear subspaces of V with $\dim F_i = i$. A *partition* is a weakly decreasing sequence of integers $\lambda\colon m \geq \lambda_1 \geq \cdots \geq \lambda_m \geq 0$. A flag $F_\bullet$ and a partition λ determine a Schubert subvariety of $\mathrm{Gr}(m,V)$,

$$X_\lambda F_\bullet \ :=\ \{H\in \mathrm{Gr}(m,V)\ \mid\ \dim H\cap F_{m+i-\lambda_i}\geq i\ \text{ for } i=1,\ldots,m\}\,.$$

This has codimension $|\lambda| := \lambda_1+\cdots+\lambda_m$ in $\mathrm{Gr}(m,V)$.

Let $\boldsymbol{\lambda}=(\lambda^1,\ldots,\lambda^s)$ be a list of partitions and $F^1_\bullet,\ldots,F^s_\bullet$ be general flags. By Kleiman's Transversality Theorem [12] the intersection

$$X_{\lambda^1}F^1_\bullet\ \cap\ X_{\lambda^2}F^2_\bullet\ \cap\ \cdots\ \cap\ X_{\lambda^s}F^s_\bullet\,. \tag{1.1}$$

is either empty or has dimension $\dim \mathrm{Gr}(m,V)-|\lambda^1|-\cdots-|\lambda^s|$. Call $\boldsymbol{\lambda}$ a *Schubert problem* if this expected dimension is zero so that (1.1) is either empty or consists of finitely many points. The number of points $d(\boldsymbol{\lambda})$ in (1.1) is independent of the choice of general flags. We will assume that $d(\boldsymbol{\lambda})\neq 0$. A choice of flags is an *instance* of the Schubert problem $\boldsymbol{\lambda}$; its *solutions* are the points in (1.1). The instance is *real* if for all i, there is some j with $\overline{F^i_\bullet}=F^j_\bullet$ and $\lambda^i=\lambda^j$, for then (1.1) is stable under complex conjugation.

Remark 1. Osculating flags provide a rich source of isotropic flags. As explained in Section 3 of [8], a rational normal curve $\gamma\colon\mathbb{C}\to V$ induces a symplectic form on V and a symplectic form on V gives rise to a rational normal curve, and we may assume that γ is real in that $\gamma(\bar{t})=\overline{\gamma(t)}$. If γ is a rational normal curve corresponding to the symplectic form $\langle\,,\,\rangle$, then every osculating flag is isotropic. (For $t\in\mathbb{C}$, the *osculating flag* $F_\bullet(t)$ is the flag whose i-plane $F_i(t)$ is spanned by $\gamma(t)$ and its derivatives $\gamma'(t),\ldots,\gamma^{(i-1)}(t)$.)

The study of real solutions to Schubert problems given by flags osculating at real points in Grassmannians and flag manifolds has been quite rich and fruitful [4, 13, 15, 16, 19, 21].

A partition λ is represented by its Young diagram, which is a left-justified array of boxes with λ_i boxes in row i. We display some partitions with their Young diagrams,

$(2,1,1)\ \longleftrightarrow$ , $\quad (2,2)\ \longleftrightarrow$ , $\quad$ and $\quad (3,2,1)\ \longleftrightarrow$ .

A partition λ is *symmetric* if it is symmetric about its main diagonal, that is, if $\lambda=\lambda'$, where λ' is the transpose of λ. The partitions $(2,2)$ and $(3,2,1)$ are symmetric while $(2,1,1)$ is not. A Schubert problem $\boldsymbol{\lambda}$ is *symmetric* if every partition in $\boldsymbol{\lambda}$ is symmetric.

Recall that our vector space V was equipped with a nondegenerate alternating bilinear form $\langle\,,\,\rangle$. A linear subspace W of V has annihilator $\angle(W)$ under $\langle\,,\,\rangle$,

$$\angle(W)\ :=\ \{v\in V\ \mid\ \langle v,w\rangle = 0 \text{ for all } w\in W\}\,,$$

and we have $\dim W+\dim\angle(W)=2m$. This induces a map $H\mapsto\angle(H)$ on $\mathrm{Gr}(m,V)$ called the *Lagrangian involution*. Given a flag $F_\bullet$, we get the flag $\angle(F_\bullet)$ whose i-plane is $\angle(F_{2m-i})$. A flag $F_\bullet$ is *isotropic* if $\angle(F_\bullet)=F_\bullet$.

The *length*, $\ell(\lambda)$ of a symmetric partition is the number of boxes on its main diagonal, so $\ell(2,2)=2$ while $\ell(2,1)=1$. We state our main theorem.

Theorem 2. *Suppose that $\boldsymbol{\lambda}=(\lambda^1,\dots,\lambda^s)$ is a symmetric Schubert problem on $\mathrm{Gr}(m,V)$ and that $F_\bullet^1,\dots,F_\bullet^s$ are isotropic flags defining a real instance of the Schubert problem $\boldsymbol{\lambda}$ such that (1.1) is finite. If $\sum_i \ell(\lambda^i)\geq m+4$, then the number (counted with multiplicity) of real points in (1.1) is congruent to the number $d(\boldsymbol{\lambda})$ of complex points, modulo four.*

Remark 3. We show in Remark 11 that $\sum_i \ell(\lambda^i)\geq m$ and this sum has the same parity as m, so that the condition in Theorem 2 for this congruence modulo four is that $\sum_i \ell(\lambda^i)$ is not equal to m or to $m+2$, which is very mild.

We use the observation of Remark 1 that osculating flags are isotropic to deduce a corollary about Schubert problems given by osculating flags. Fix a real rational normal curve $\gamma\colon \mathbb{C}\to V$ with corresponding osculating flags $F_\bullet(t)$ for $t\in\mathbb{C}$, and a symmetric Schubert problem $\boldsymbol{\lambda}=(\lambda^1,\dots,\lambda^s)$ on $\mathrm{Gr}(m,V)$. An *osculating instance* of this Schubert problem is a list of distinct complex numbers $t_1,\dots,t_s$ which give corresponding osculating flags $F_\bullet(t_1),\dots,F_\bullet(t_s)$. This osculating instance is *real* if the set $\{t_1,\dots,t_s\}$ is closed under complex conjugation and if $\overline{t_i}=t_j$ implies that $\lambda_i=\lambda_j$.

We deduce a corollary to Theorem 2 that implies Conjecture 21 of [8], which was the strongest result one could reasonably expect to hold concerning this congruence modulo four for symmetric Schubert problems. This is strictly stronger than all congruence results obtained in [8].

Corollary 4. *Suppose that $\boldsymbol{\lambda}=(\lambda^1,\dots,\lambda^s)$ is a symmetric Schubert problem on $\mathrm{Gr}(m,V)$ and that $F_\bullet(t_1),\dots,F_\bullet(t_s)$ are osculating flags defining a real instance of $\boldsymbol{\lambda}$. If $\sum_i \ell(\lambda^i)\geq m+4$, then the number (counted with multiplicity) of real points in*

$$X_{\lambda^1}F_\bullet(t_1)\,\cap\, X_{\lambda^2}F_\bullet(t_2)\,\cap\,\cdots\,\cap\, X_{\lambda^s}F_\bullet(t_s)\,.$$

is congruent to the number $d(\boldsymbol{\lambda})$ of complex points, modulo four.

We do not need to assert that the intersection consists of finitely many points, for it always does [2].

Remark 5. When $d(\boldsymbol{\lambda})$ is congruent to two modulo four and $\sum_i \ell(\lambda^i)\geq m+4$, there will always be at least two real solutions to a real instance of the symmetric Schubert problem $\boldsymbol{\lambda}$ given by isotropic flags. Such lower bounds implied by Theorem 2 occur frequently. Table 1 gives the total number of symmetric Schubert problems in $\mathrm{Gr}(m,V)$ for $m=2,\dots,7$, together with the number of those for which Theorem 2 implies a lower bound of two real solutions to a real instance given by isotropic flags. Timings are reported in GHz-seconds (s), GHz-hours (h), and GHz-years (y).

Real instances of nine of the smallest problems from Table 1 involving osculating flags were computed in [7]. Few other problems from this table are feasible to compute using symbolic methods. For eight (numbers 20, 57, 507, 568, 586, 587, 590, and 595 from [7]) at least one real instance had only two real solutions, showing that the lower bound of two is sharp in these cases.

Table 1. Numbers of symmetric Schubert problem with a lower bound of two.

m	2	3	4	5	6	7
Symmetric	1	8	81	1037	16933	349844
Lower bound by Theorem 2	0	2	14	199	3289	82753
Percentage	0	25	17.3	19.2	19.4	23.7
Time	0.57 s	0.6 s	1.9 s	158 s	17.1 h	1.15 y

Real osculating instances of problem number 773 from [7] with 70 solutions in $\mathrm{Gr}(5,10)$ have a sharp lower bound of six real solutions. This is a member of a family of such symmetric Schubert problems. By results in [9, § 3], when $m = 2k{+}1$ is odd, a real osculating instance of the symmetric problem on $\mathrm{Gr}(2k{+}1, V)$ given by one condition $(2k,\dots,2k,0)$ and $4k{-}1$ conditions $\square$, which has $\binom{4k}{2k}$ complex solutions, will have at least $\binom{2k}{k}$ real solutions, and this lower bound is attained.

2. Proof of Theorem 2

We follow the main line of argument for the results of [8]. We observe that the Lagrangian involution $H \mapsto \angle(H)$ permutes the solutions to an instance of a symmetric Schubert problem $\boldsymbol{\lambda}$ given by isotropic flags and then construct a family $\mathcal{X}_{\boldsymbol{\lambda}} \to \mathcal{Z}_{\boldsymbol{\lambda}}$ whose base parameterizes instances of the Schubert problem $\boldsymbol{\lambda}$ given by isotropic flags and whose fibers are the solutions to those instances. We then estimate the codimension of the $\angle$-fixed point locus of the family $\mathcal{X}_{\boldsymbol{\lambda}} \to \mathcal{Z}_{\boldsymbol{\lambda}}$, which shows that the numerical condition $\sum_i \ell(\lambda^i) \geq m{+}4$ implies that the fixed points have codimension at least two. Finally, we invoke a key lemma from [8] to complete the proof.

2.1. The Lagrangian Grassmannian.

An m-dimensional subspace H of V is *Lagrangian* if $\angle(H) = H$. The set of all Lagrangian subspaces of V forms the *Lagrangian Grassmannian* $\mathrm{LG}(V)$. This is smooth of dimension $\binom{m+1}{2}$ and is a homogeneous space for the symplectic group $\mathrm{Sp}(V)$ of linear transformations of V which preserve $\langle\,,\,\rangle$.

An isotropic flag $F_\bullet$ and a symmetric partition λ determine a Schubert subvariety $Y_\lambda F_\bullet$ of $\mathrm{LG}(V)$, which is the intersection $X_\lambda F_\bullet \cap \mathrm{LG}(V)$,

$$Y_\lambda F_\bullet \ := \ \{H \in \mathrm{LG}(V) \ \mid \ \dim H \cap F_{m+i-\lambda_i} \geq i \ \text{ for } i = 1,\dots,m\}\,.$$

This has codimension $\|\lambda\| := \frac{1}{2}(|\lambda| + \ell(\lambda))$ in $\mathrm{LG}(V)$.

We need the following result which partially explains why these Lagrangian Schubert varieties are relevant for Theorem 2.

Proposition 6 (Lemma 9 of [8]). *Let λ be a partition and $F_\bullet$ a flag. Then*

$$\angle(X_\lambda F_\bullet) \ = \ X_{\lambda'} \angle(F_\bullet)\,.$$

Thus if λ is symmetric and $F_\bullet$ isotropic, then $\angle(X_\lambda F_\bullet) = X_\lambda F_\bullet$ and $Y_\lambda = (X_\lambda F_\bullet)^\angle$, the set of points of $X_\lambda F_\bullet$ that are fixed by $\angle$. This has the following consequence.

Corollary 7. *The Lagrangian involution permutes the solutions to a symmetric Schubert problem given by isotropic flags.*

2.2. **Families associated to Schubert problems.** Let $\boldsymbol{\lambda}$ be a symmetric Schubert problem. We construct families whose bases parameterize all instances of $\boldsymbol{\lambda}$ given by isotropic flags and whose fibers are the solutions to the corresponding instance.

The set $\mathbb{F}\ell$ of isotropic flags in V is a flag manifold for $\mathrm{Sp}(V)$ of dimension m^2. Define

$$\mathcal{U}^*_{\boldsymbol{\lambda}} \;:=\; \{(F^1_\bullet,\dots,F^s_\bullet,\, H) \mid F^i_\bullet \in \mathbb{F}\ell \;\text{ and }\; H \in X_{\lambda^i}F^i_\bullet \;\text{ for } i=1,\dots,s\}\,.$$

We have the two projections

$$\pi \,:\, \mathcal{U}^*_{\boldsymbol{\lambda}} \;\longrightarrow\; (\mathbb{F}\ell)^s \qquad\text{and}\qquad \mathrm{pr} \,:\, \mathcal{U}^*_{\boldsymbol{\lambda}} \;\longrightarrow\; \mathrm{Gr}(m,V)\,.$$

For isotropic flags $F^1_\bullet,\dots,F^s_\bullet$, the fiber $\mathrm{pr}(\pi^{-1}(F^1_\bullet,\dots,F^s_\bullet))$ consists of the solutions

$$X_{\lambda^1}F^1_\bullet \,\cap\, X_{\lambda^2}F^2_\bullet \,\cap\, \cdots \,\cap\, X_{\lambda^s}F^s_\bullet \tag{2.1}$$

to the instance of the Schubert problem $\boldsymbol{\lambda}$ given by the flags $F^1_\bullet,\dots,F^s_\bullet$.

As $\mathrm{Sp}(V)$ does not act transitively on $\mathrm{Gr}(m,V)$, we cannot use Kleiman's Theorem [12] to conclude that an intersection (2.1) given by general flags is transverse. Transversality follows instead from the main result of [23]. Consequently, there is a nonempty Zariski open subset $\mathcal{O} \subset (\mathbb{F}\ell)^s$ consisting of s-tuples of isotropic flags for which the intersection (2.1) is transverse and therefore consists of $d(\boldsymbol{\lambda})$ points.

We seek a family $\mathcal{X} \to \mathcal{Z}$ of instances of $\boldsymbol{\lambda}$ where $\dim\mathcal{X} = \dim\mathcal{Z}$ and $\mathcal{Z}$ is irreducible with $\mathcal{Z}(\mathbb{R})$ parameterizing all real instances of $\boldsymbol{\lambda}$. Since we cannot easily compute the dimension of $\mathcal{U}^*_{\boldsymbol{\lambda}}$, we replace it by a possibly smaller set. Define $\mathcal{U}_{\boldsymbol{\lambda}}$ to be the closure of $\pi^{-1}(\mathcal{O})$ in $\mathcal{U}^*_{\boldsymbol{\lambda}}$. Restricting π to $\mathcal{U}_{\boldsymbol{\lambda}}$ gives the dominant map

$$\pi \,:\, \mathcal{U}_{\boldsymbol{\lambda}} \;\longrightarrow\; (\mathbb{F}\ell)^s\,, \tag{2.2}$$

where a fiber $\pi^{-1}(F^1_\bullet,\dots,F^s_\bullet)$ is a subset of the intersection (2.1) and is equal to it when the intersection is finite. Thus $\dim\mathcal{U}_{\boldsymbol{\lambda}} = \dim(\mathbb{F}\ell)^s = s\cdot m^2$.

This family (2.2) has the fault that the real points of its base $(\mathbb{F}\ell)^s$ are s-tuples of real isotropic flags, which are only some of the flags giving real instances of $\boldsymbol{\lambda}$.

Let $S_{\boldsymbol{\lambda}} \subset S_s$ be the group of permutations σ of $\{1,2,\dots,s\}$ with $\lambda^i = \lambda^{\sigma(i)}$ for all $i=1,\dots,s$. Then $S_{\boldsymbol{\lambda}} \simeq S_{a_1}\times\cdots\times S_{a_t}$ where $\boldsymbol{\lambda}$ consists of t distinct partitions $\mu^1,\dots,\mu^t$ with μ^i occurring a_i times. Then $S_{\boldsymbol{\lambda}}$ acts on the families $\mathcal{U}^*_{\boldsymbol{\lambda}}, \mathcal{U}_{\boldsymbol{\lambda}} \to (\mathbb{F}\ell)^s$, preserving fibers,

$$\mathrm{pr}(\pi^{-1}(F^1_\bullet,\dots,F^s_\bullet)) \;=\; \mathrm{pr}(\pi^{-1}(F^{\sigma(1)}_\bullet,\dots,F^{\sigma(s)}_\bullet)) \qquad\text{for all } \sigma\in S_{\boldsymbol{\lambda}}\,.$$

Define $\pi\colon \mathcal{X}_{\boldsymbol{\lambda}} \to \mathcal{Z}_{\boldsymbol{\lambda}}$ to be the quotient of $\mathcal{U}_{\boldsymbol{\lambda}} \to (\mathbb{F}\ell)^s$ by the group $S_{\boldsymbol{\lambda}}$.

2.3. **Proof of Theorem 2.** We defer the proof of the following lemma.

Lemma 8. *The map $\pi\colon \mathcal{X}_{\boldsymbol{\lambda}} \to \mathcal{Z}_{\boldsymbol{\lambda}}$ is a proper dominant map of real varieties of the same dimension with $\mathcal{Z}_{\boldsymbol{\lambda}}$ smooth and $\mathcal{Z}_{\boldsymbol{\lambda}}(\mathbb{R})$ connected. The Lagrangian involution preserves fibers of π and the codimension in $\mathcal{X}_{\boldsymbol{\lambda}}$ of the $\angle$-fixed points $\mathcal{X}^{\angle}_{\boldsymbol{\lambda}}$ is at least $\frac{1}{2}(\sum_i \ell(\lambda^i) - m)$.*

We recall Lemma 5 from [8].

Proposition 9. *Let $f\colon X \to Z$ be a proper dominant map of real varieties of the same dimension with Z smooth. Suppose that X has an involution $\angle$ preserving the fibers of f such that the image in Z of the set of $\angle$-fixed points has codimension at least 2.*

If $y, z \in Z(\mathbb{R})$ belong to the same connected component of $Z(\mathbb{R})$, the fibers above them are finite and at least one contains no $\angle$-fixed points, then we have

$$\# f^{-1}(y) \cap X(\mathbb{R}) \;\equiv\; \# f^{-1}(z) \cap X(\mathbb{R}) \mod 4\,.$$

Remark 10. Lemma 5 in [8] requires that there are no $\angle$-fixed points in either fiber $\pi^{-1}(y)$ or $\pi^{-1}(z)$. This may be relaxed to only one fiber avoiding $\angle$-fixed points, which may be seen using a limiting argument along the lines of the proof of Corollary 7 in [8].

Proof of Theorem 2. By Lemma 8, the hypotheses of Proposition 9 hold, as the inequality $\sum_i \ell(\lambda^i) \geq m+4$ implies that $\operatorname{codim} \pi(\mathcal{X}^{\angle}_{\boldsymbol{\lambda}}) \geq \operatorname{codim}(\mathcal{X}^{\angle}_{\boldsymbol{\lambda}}) \geq 2$. Let $(F^1_\bullet, \dots, F^s_\bullet)$ be isotropic flags defining a real instance of the Schubert problem $\boldsymbol{\lambda}$ such that (2.1) is finite.

Since this instance is real, for each $i = 1, \dots, s$ if $\overline{F^i_\bullet} = F^j_\bullet$, then $\lambda^i = \lambda^j$. Thus there is a permutation $\sigma \in S_{\boldsymbol{\lambda}}$ such that $\overline{F^i_\bullet} = F^{\sigma(i)}_\bullet$ for $i = 1, \dots, s$, and so the image of $(F^1_\bullet, \dots, F^s_\bullet)$ in $\mathcal{Z}_{\boldsymbol{\lambda}}$ is a real point $y \in \mathcal{Z}_{\boldsymbol{\lambda}}(\mathbb{R})$. We complete the proof by exhibiting a point $z \in \mathcal{Z}_{\boldsymbol{\lambda}}(\mathbb{R})$ for which $\pi^{-1}(z)$ consists of $d(\boldsymbol{\lambda})$ real points, none of which are fixed by $\angle$.

For distinct $t_1, \dots, t_s \in \mathbb{R}$, the intersection

$$X_{\lambda^1} F_\bullet(t_1) \cap X_{\lambda^2} F_\bullet(t_2) \cap \cdots \cap X_{\lambda^s} F_\bullet(t_s) \tag{2.3}$$

is transverse and consists of $d(\boldsymbol{\lambda})$ real points, by the Mukhin-Tarasov-Varchenko Theorem [15]. The osculating flags $F_\bullet(t_i)$ are real and isotropic, and we would be done if there were no $\angle$-fixed points in (2.3). This is equivalent to the intersection of the corresponding Lagrangian Schubert varieties being empty. This is unknown, but expected, as it follows from Conjecture 5.1 in [22] which is supported by significant evidence.

Since the intersection (2.3) is transverse, if $(E^1_\bullet, \dots, E^s_\bullet) \in (\mathbb{F}\ell)^s$ are real isotropic flags that are sufficiently close to the osculating flags in (2.3), then the intersection

$$X_{\lambda^1} E^1_\bullet \cap X_{\lambda^2} E^2_\bullet \cap \cdots \cap X_{\lambda^s} E^s_\bullet \tag{2.4}$$

is transverse and consists of $d(\boldsymbol{\lambda})$ real points. By Kleiman's Theorem [12] we may also assume that $(E^1_\bullet, \dots, E^s_\bullet)$ are general in that the intersection

$$Y_{\lambda^1} E^1_\bullet \cap Y_{\lambda^2} E^2_\bullet \cap \cdots \cap Y_{\lambda^s} E^s_\bullet \tag{2.5}$$

of Lagrangian Schubert varieties is either empty or has dimension

$$\begin{aligned}\binom{m+1}{2} - \sum_{i=1}^{s} \|\lambda^i\| \;&=\; \binom{m+1}{2} - \frac{1}{2}\sum_{i=1}^{s} |\lambda^i| - \frac{1}{2}\sum_{i=1}^{s} \ell(\lambda^i) \\ &\leq\; \frac{m^2}{2} + \frac{m}{2} - \frac{m^2}{2} - \frac{m}{2} - 2 \;=\; -2\,.\end{aligned}$$

We conclude that (2.5) is empty and therefore (2.4) contains no Lagrangian subspaces.

If $z \in \mathcal{Z}_{\boldsymbol{\lambda}}(\mathbb{R})$ is the image of $(E^1_\bullet, \dots, E^s_\bullet) \in (\mathbb{F}\ell)^s$, then the fiber $\pi^{-1}(z)$ (which is (2.4)) consists of $d(\boldsymbol{\lambda})$ real points, none of which are Lagrangian. This completes the proof. $\square$

Proof of Lemma 8. Consider the quotient of $(\mathbb{F}\ell)^s$ by the group $S_{\boldsymbol{\lambda}}$, which is the product

$$\mathcal{Z}_{\boldsymbol{\lambda}} \;=\; \operatorname{Sym}_{a_1}(\mathbb{F}\ell) \times \operatorname{Sym}_{a_2}(\mathbb{F}\ell) \times \cdots \times \operatorname{Sym}_{a_t}(\mathbb{F}\ell)\,,$$

where $\mathrm{Sym}_a(\mathbb{F}\ell)$ is the quotient $(\mathbb{F}\ell)^a/S_a$ and $\boldsymbol{\lambda}$ consists of t distinct partitions $\mu^1,\dots,\mu^t$ with μ^i occurring a_i times in $\boldsymbol{\lambda}$.

For $F_\bullet\in\mathbb{F}\ell$, let $Z^\circ_e F_\bullet\subset\mathbb{F}\ell$ be those flags in linear general position with respect to $F_\bullet$. This dense subset of $\mathbb{F}\ell$ is a Schubert variety isomorphic to $\mathbb{C}^{m^2}$. As $F_\bullet$ varies in $\mathbb{F}\ell$, these form an affine cover of $\mathbb{F}\ell$. Given a finite set $\{F^1_\bullet,\dots,F^a_\bullet\}$ of isotropic flags, there is an isotropic flag $F_\bullet$ that is simultaneously in linear general position with each $F^i_\bullet$, so that $\{F^1_\bullet,\dots,F^a_\bullet\}\subset Z^\circ_e F_\bullet$. Thus $(\mathbb{F}\ell)^a$ is covered by the S_a-invariant affine varieties $(Z^\circ_e F_\bullet)^a$, each isomorphic to $(\mathbb{C}^{m^2})^a$. By descent, this implies that the quotient $\mathrm{Sym}_a(\mathbb{F}\ell)=(\mathbb{F}\ell)^a/S_a$ is well-defined and covered by affine varieties $(Z^\circ_e F_\bullet)^a/S_a$, each isomorphic to $(\mathbb{C}^{m^2})^a/S_a\simeq(\mathbb{C}^{m^2})^a$, as $\mathbb{C}^a/S_a\simeq\mathbb{C}^a$. It follows that $\mathrm{Sym}_a(\mathbb{F}\ell)$ is a smooth irreducible variety whose real points are connected which implies the same for $\mathcal{Z}_{\boldsymbol{\lambda}}$.

The map $\pi\colon\mathcal{U}^*_{\boldsymbol{\lambda}}\to(\mathbb{F}\ell)^s$ is proper as it comes from a projection along a Grassmannian factor. Its fibers are preserved by the Lagrangian involution and are equal over points in an $S_{\boldsymbol{\lambda}}$-orbit. Both properties hold for $\pi^{-1}(\mathcal{O})\to\mathcal{O}$ (as $\mathcal{O}$ is $S_{\boldsymbol{\lambda}}$-stable) and therefore for $\pi\colon\mathcal{U}_{\boldsymbol{\lambda}}\to(\mathbb{F}\ell)^s$. We conclude that π descends to the quotient $\pi\colon\mathcal{X}_{\boldsymbol{\lambda}}\to\mathcal{Z}_{\boldsymbol{\lambda}}$, where it is a proper dominant map and the Lagrangian involution preserves its fibers.

Since $\dim\mathcal{U}_{\boldsymbol{\lambda}}=\dim(\mathbb{F}\ell)^s=s\cdot m^2$ and $S_{\boldsymbol{\lambda}}$ is a finite group, we conclude that $\dim\mathcal{X}_{\boldsymbol{\lambda}}=\dim\mathcal{Z}_{\boldsymbol{\lambda}}=s\cdot m^2$.

We study the $\angle$-fixed points of $\mathcal{U}^*_{\boldsymbol{\lambda}}$ which form the universal family,

$$\mathcal{L}_{\boldsymbol{\lambda}}\ :=\ \{(F^1_\bullet,\dots,F^s_\bullet,H)\ \mid\ F^i_\bullet\in\mathbb{F}\ell\ \text{ and }\ H\in Y_{\lambda^i}F^i_\bullet\ \text{ for } i=1,\dots,s\}\,.$$

Consider the projection $\mathrm{pr}\colon\mathcal{L}_{\boldsymbol{\lambda}}\to\mathrm{LG}(V)$. Let $H\in\mathrm{LG}(V)$. Then

$$\begin{aligned}\mathrm{pr}^{-1}(H)\ &=\ \{(F^1_\bullet,\dots,F^s_\bullet,H)\ \mid\ H\in Y_{\lambda^i}F^i_\bullet\ \text{ for } i=1,\dots,s\}\\ &\simeq\ \prod_{i=1}^{s}\{F_\bullet\in\mathbb{F}\ell\ \mid\ H\in Y_{\lambda^i}F_\bullet\}\,.\end{aligned}$$

For λ symmetric and $H\in\mathrm{LG}(V)$, define

$$Z_\lambda(H)\ :=\ \{F_\bullet\in\mathbb{F}\ell\ \mid\ H\in Y_\lambda F_\bullet\}\,.$$

This is a Schubert subvariety of $\mathbb{F}\ell$ of codimension $\|\lambda\|$. Thus

$$\mathrm{pr}^{-1}(H)\ =\ Z_{\lambda^1}(H)\ \times\ Z_{\lambda^2}(H)\ \times\ \cdots\ \times\ Z_{\lambda^s}(H)\,,$$

which has codimension $\sum_i\|\lambda^i\|=\frac{1}{2}\sum_i\bigl(|\lambda^i|+\ell(\lambda^i)\bigr)$ in $(\mathbb{F}\ell)^s$ and is irreducible as each $Z_{\lambda^i}(H)$ is a Schubert variety and is therefore irreducible. Thus $\mathrm{pr}\colon\mathcal{L}_{\boldsymbol{\lambda}}\to\mathrm{LG}(V)$ exhibits $\mathcal{L}_{\boldsymbol{\lambda}}$ as a fiber bundle. We compute its dimension,

$$\begin{aligned}\dim\mathcal{L}_{\boldsymbol{\lambda}}\ &=\ \dim\mathrm{LG}(V)\ +\ \dim\mathrm{pr}^{-1}(H)\ =\ \binom{m+1}{2}\ +\ s\cdot m^2\ -\ \sum_{i=1}^{s}\|\lambda^i\|\\ &=\ s\cdot m^2\ -\ \frac{1}{2}\Bigl(\sum_{i=1}^{s}\ell(\lambda^i)-m\Bigr)\,.\end{aligned}$$

Thus $\dim\mathcal{U}_{\boldsymbol{\lambda}}\cap\mathcal{L}_{\boldsymbol{\lambda}}\le s\cdot m^2-\frac{1}{2}\bigl(\sum_i\ell(\lambda^i)-m\bigr)$. As $\mathcal{U}_{\boldsymbol{\lambda}}\cap\mathcal{L}_{\boldsymbol{\lambda}}$ is the set of $\angle$-fixed points of $\mathcal{U}_{\boldsymbol{\lambda}}$, $\dim\mathcal{U}_{\boldsymbol{\lambda}}=s\cdot m^2$, and $\mathcal{X}_{\boldsymbol{\lambda}}$ is the quotient of $\mathcal{U}_{\boldsymbol{\lambda}}$ by the finite group $S_{\boldsymbol{\lambda}}$, the $\angle$-fixed points in $\mathcal{X}_{\boldsymbol{\lambda}}$ have codimension at least $\frac{1}{2}\bigl(\sum_i\ell(\lambda^i)-m\bigr)$. $\square$

Remark 11. If $\boldsymbol{\lambda}$ is a symmetric Schubert problem, the quantity

$$\sum_{i=1}^{s} \|\lambda^i\| \;=\; \frac{1}{2}\sum_{i=1}^{s}(|\lambda^i| + \ell(\lambda^i)) \;=\; \frac{m^2}{2} + \frac{1}{2}\sum_{i=1}^{s}\ell(\lambda^i)$$

is an integer, so $\sum_i \ell(\lambda^i)$ has the same parity as m. For generic flags $(E^1_\bullet, \ldots, E^s_\bullet)$, the intersection (2.5) of Lagrangian Schubert varieties is a subset of the intersection (2.4) of Schubert varieties. By Kleiman's Theorem, this gives the inequality

$$\binom{m+1}{2} - \sum_{i=1}^{s} \|\lambda^i\| \;\leq\; m^2 - \sum_{i=1}^{s} |\lambda^i|\,,$$

which implies that $m \leq \sum_i \ell(\lambda^i)$. Thus the only possibilities for $\sum_i \ell(\lambda^i)$ for which Theorem 2 does not imply a congruence modulo four are m or $m{+}2$.

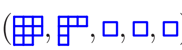
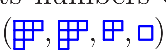
When $\sum_i \ell(\lambda^i) = m$, we have $\binom{m+1}{2} = \sum_i \|\lambda^i\|$ so that $\boldsymbol{\lambda}$ is a Schubert problem for $\mathrm{LG}(V)$ with $c(\boldsymbol{\lambda})$ solutions. That is, for general isotropic flags $E^1_\bullet, \ldots, E^s_\bullet$ the intersection (2.5) is transverse and consists of $c(\boldsymbol{\lambda})$ points. When $c(\boldsymbol{\lambda}) \neq 0$ the family $\mathcal{U}_{\boldsymbol{\lambda}} \to (\mathbb{F}\ell)^s$ is reducible: $\mathcal{L}_{\boldsymbol{\lambda}}$ is one component and $\overline{\mathcal{U}_{\boldsymbol{\lambda}} \smallsetminus \mathcal{L}_{\boldsymbol{\lambda}}}$ is the other.

For example, the problem () on $\mathrm{Gr}(4,8)$ with eight solutions has $\sum_i \ell(\lambda^i) = 4 = m$. The corresponding problem in $\mathrm{LG}(\mathbb{C}^8)$ has four solutions. Thus four of the eight solutions on $\mathrm{Gr}(4,8)$ will be isotropic and the other four will not be isotropic. In our experimentation (number 490 on [7]) this problem exhibits a congruence modulo four.

When $\sum_i \ell(\lambda^i) = m+2$, a general intersection (2.5) of Lagrangian Schubert varieties is empty and $\pi^{-1}(\mathcal{O})$ does not meet $\mathcal{L}_{\boldsymbol{\lambda}}$. There are three possibilities.

(1) $\mathcal{L}_{\boldsymbol{\lambda}} \subset \mathcal{U}_{\boldsymbol{\lambda}}$ and $\pi\colon \mathcal{L}_{\boldsymbol{\lambda}} \to \pi(\mathcal{L}_{\boldsymbol{\lambda}})$ generically has finite fibers.
(2) $\mathcal{L}_{\boldsymbol{\lambda}} \subset \mathcal{U}_{\boldsymbol{\lambda}}$ and $\pi\colon \mathcal{L}_{\boldsymbol{\lambda}} \to \pi(\mathcal{L}_{\boldsymbol{\lambda}})$ has positive dimensional fibers.
(3) $\mathcal{L}_{\boldsymbol{\lambda}} \not\subset \mathcal{U}_{\boldsymbol{\lambda}}$.

In case (1), $\pi(\mathcal{L}_{\boldsymbol{\lambda}})$ has codimension one as does the image of the set of $\angle$-fixed points of $\mathcal{X}_{\boldsymbol{\lambda}}$, so Proposition 9 does not necessarily imply a congruence modulo four. In cases (2) and (3), $\pi(\mathcal{L}_{\boldsymbol{\lambda}})$ has codimension two, and so there will be a congeuence modulo four.

We have observed some symmetric Schubert problems with $\sum_i \ell(\lambda^i) = m+2$ that have a congruence modulo four and some that do not (so that (1) holds). For example, the problem () on $\mathrm{Gr}(4,8)$ with six solutions has $\sum_i \ell(\lambda^i) = 6 = m{+}2$ (number 495 on [7]) and its numbers of real solutions do not exhibit a congruence modulo four, but the problem () on $\mathrm{Gr}(4,8)$ with eight solutions has $\sum_i \ell(\lambda^i) = 6 = m{+}2$ (number 497 on [7]) and its possible numbers of real solutions appear to be congruent modulo four. We believe that (3) is unlikely and that (2) holds if and only if there is a congruence modulo four.

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